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UNIVERSITY COLLEGE,
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*WITH 231 ILLUSTRATIONS*

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# PREFACE

THIS book is intended for the use of students in Advanced Science Classes. It has been written with the object of giving an account of the chief points of the theory of Optics, both Geometrical and Physical, without making great demands from students in the way of Mathematical attainments. I have made some very elementary applications of the Differential Calculus; but have, in general, given as well alternative methods, or have indicated how the results could be otherwise obtained; and have, I hope, by occasionally presenting the two methods of solution side by side, done something to encourage students in the use of the shorter and more elegant one. The entire book may thus be read without any more advanced knowledge of Mathematics than Higher Trigonometry.

I have throughout kept in view the experimental side of the subject, and have given some account of the methods of making experiments and optical measurements, as well as indicating experiments that can be performed sometimes by very simple means in illustration of important phenomena.

The majority of the illustrations have been engraved from my drawings. For others I am indebted to various books, and in particular to Glazebrook's "Physical Optics" and Jamin's "Cours de Physique." I have also to thank Messrs. Nalder Bros. and Co. for the illustration of their optical bench.

<div style="text-align:right">W. T. A. EMTAGE.</div>

LONDON,
  *September,* 1896.

# CONTENTS

# LIGHT.

## CHAPTER I.

### *RECTILINEAR PROPAGATION OF LIGHT.*

For an object to be seen, there must be light by which to see it. Light is an action taking place in the space between the object seen and the eye which sees it; and which must also, as a rule, exist independently of the object. For most bodies are seen only by means of the modifications which they produce in light; only those which are self-luminous—that is, which could be seen if taken into a dark room—are themselves the sources of the light by which they are seen. As a rule, a body is seen through modifying in various ways light coming from an independent source, as, for instance, the sun or a candle; and, by means of the light so modified and then reaching our eyes, we are able to form a judgment of the body with regard to form, colour, size, and distance. A large part of the study of light consists in considering the various ways in which it is modified by different bodies, either in passing off from the body when it has fallen on its surface, or in passing into it if the body is transparent—that is, if it allows light to pass through it and come out again as light.

Whatever be the nature of the action which we call light, there are certain simple geometrical laws which this action obeys, and the study of which will not depend on the physical nature of the action. The study of these laws will lead to many important results, as, for instance, those which relate to optical instruments; and will explain many optical phenomena, such as the chief phenomena of the rainbow. This part of the study is called **Geometrical Optics.**

The study of the phenomena which indicate the physical nature of light is called **Physical Optics.**

In the geometrical study of light the first thing that we

must notice is its passage in straight lines. In any homogeneous medium, as in a vacuum or in homogeneous atmospheric air, the action by which a small body or single point, as we may call it, of an object is seen at a given point takes place in the straight line joining the body to the point. We may show this by means of the following experiment :—

Take three screens, with small holes pierced in them. Adjust the three holes in a straight line by any means, such as by getting a stretched thread to pass through the three without being constrained by the edges of the middle hole. Light may now be seen to pass through the three holes ; but if any one of the screens is displaced, the light is cut off. Again, if the screens are arranged as diaphragms in the same tube, and so adjusted that light passes through the holes, then, however the tube be rotated about its axis, light will continue to pass through. But obviously, if the light has a given path in space between the extreme holes, the middle one can only be so arranged as to suit the passage of the light (however the tube be rotated) if the path is a straight line.

[So well recognized is it that light travels in straight lines, that the readiest method of getting three such apertures in a straight line would be by so arranging them that light from any source passes through all three.]

A **ray** of light, in geometrical optics, must be taken to mean simply a straight line along which light passes ; that is, any straight line drawn from the source of light, or from any visible point, is a ray.

A **pencil** is the light in an assemblage of rays in general passing through a single point ; that is, it consists geometrically of all the rays contained in a cone, generally of very small angle, having this point as its vertex. An eye sees a visible point by means of all the light in a pencil having that point as vertex, and which enters the pupil of the eye. A pencil may consist, however, of rays coming from or going to a point ; that is, it may be either **divergent** or **convergent**. Or, again, the rays may be parallel to each other ; then the pencil is called a **parallel** pencil. The vertex of the cone, or point through which all the rays pass, is called the **focus** of the pencil. But we shall have to consider pencils in which there is no single point through which all the rays pass.

A **beam** is the light in any assemblage of rays coming not necessarily from a single point ; as, for instance, the light which would come from the whole body of the sun through an aperture formed in a screen.

The words **ray** and **pencil**, in geometrical reasoning and constructions, need not be taken to have any physical significance beyond showing the directions of passage of the light. The straight line joining two apertures, which may be regarded as points, in two screens, is the line along which passes the light that is received by an eye that looks through the apertures simultaneously. This light itself is sometimes spoken of as a ray. But according to the definitions here given it would be a small beam.

Suppose a ray of light to be bent out of its course by any cause. Then by the **deviation** of the ray is meant the angle between its initial and final directions, both directions being reckoned in the sense in which the light travels. So that if, for instance, a ray is bent right back on itself, as by **reflexion**, its deviation is 180°.

Although a ray, as already said, is generally taken to be a straight line, it will frequently be convenient to speak of the path of the light all along its course, even though deviated, as a ray.

Suppose we have a point source of light illuminating a surface, say a screen. Now let an opaque object be interposed between the source and the screen, so as to cut off the light from a part only of the screen. The area from which the light is thus cut off by the object is called the shadow of the object. If we describe a cone, with its vertex at the point and its sides all touching the opaque object, this is called a **shadow-cone**, and gives, by its intersection with any surface, the shadow which the object will cast on that surface. In the case of a theoretical point source, as here considered, the edge of the shadow would be absolutely defined, all points within it being completely cut off from illumination by the source, and all points outside it being uninfluenced by the presence of the shadow-casting object. Such a theoretical shadow is called a **geometrical shadow**. As a matter of fact, any source of light is of finite extent, so that the edge of a shadow cannot be so well defined. And further, on account of the peculiar nature of light, the illumination on the screen must vary continuously, although it may vary very rapidly; that is, it is impossible to pass with absolute abruptness from points in complete shadow to points where the illumination is full. At the same time, the effect due to this cause is so small that very special means must be taken to show it, and we may, as a rule, consider the definition of the edge of the shadow as independent of this cause. Although we cannot

have a point source in practice, simple experiments will show that the more limited the source of light is, the better defined is the shadow. A very small gas flame will give a better shadow than a large one, although the large one gives much more illumination. The sharp shadows, too, given by an arc lamp may be referred to.

Now let us consider an illuminating source of finite extent, such as S, and let a shadow-casting object, O, be placed between this and a screen. By drawing the common tangent lines to
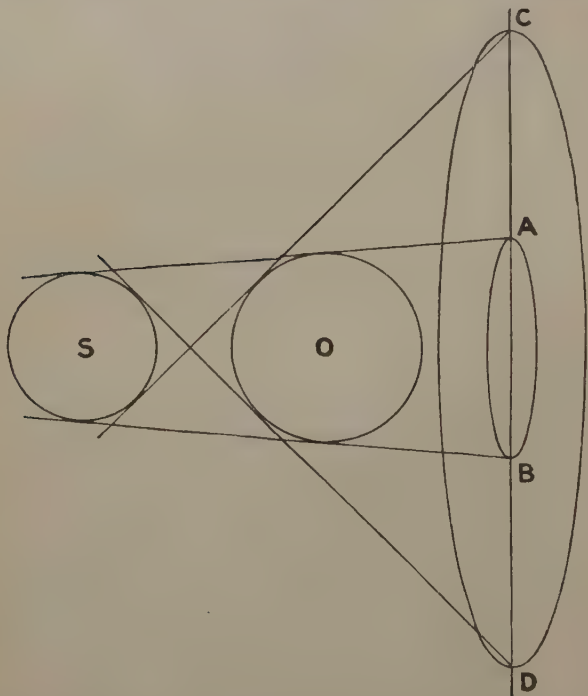


Fig. 1.

S and O, and producing them to meet the screen, we see that they divide the screen into three parts: (1) a region within the curve A B, which is illuminated by no part of S, or such as that S can be seen from no point of it; (2) a region outside the curve C D which is fully illuminated, or with the illumination of which O in no way interferes, or such that S is

fully visible from any point of it ; (3) a region between A B and C D, any point of which is only partly illuminated, O cutting off the illumination that would reach this point from a portion of S, or this region is such that S is only partly visible from any point of it. This third region is called a **penumbra.** It is clear that the illumination in it will increase uniformly as we pass outward from the boundary A B to the boundary C D.

A penumbra may exist without a shadow, as the accompanying figure shows. In this case O is too small to cast a
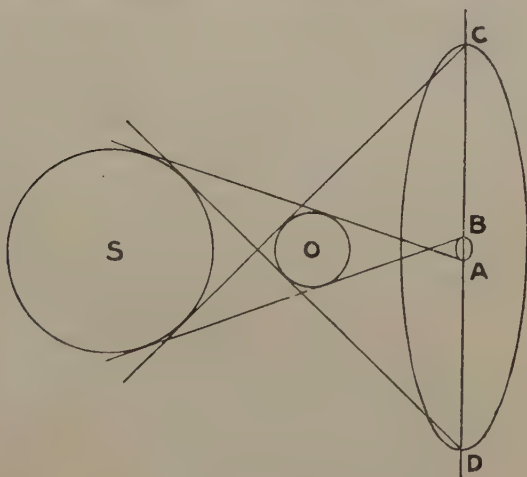


FIG. 2.

complete shadow. The region A B receives illumination from S on both sides of O. The screen could, however, be put nearer to O in such a position as to show a shadow.

**Camera Obscura.**—If a small hole is made in the shutter of a darkened room, and a screen placed so that light can fall on it through the hole, the objects outside being well lighted, an inverted image of them will be formed on the screen. For from each point without, a small pencil of light will pass through the aperture and illuminate a very small portion of the screen. The assemblage of images so formed of the outside points will be an image of the objects. This will not be very clear and well defined, because to each point there will correspond on the screen a small illuminated area of

size depending on the size of the aperture; and these areas will overlap each other, and thus produce a blurred image. The image may be made sharper by diminishing the size of the aperture; but then the amount of illumination on the screen is diminished.

A similar experiment may be made by holding a sheet of paper with a pin-hole in it between a lighted candle and a screen, when an inverted image of the candle will be produced on the screen.

**Law of Inverse Squares.**—Suppose we have a source of light, S, whose dimensions are so small compared with the other distances in question that it may be regarded as a theoretical point source. Let this illuminate a surface, A B C D, so that A B C D receives all the light coming from S in the cone S A B C D. Let now A B C D be removed and another surface, A′ B′ C′ D′, be illuminated instead of it. Let this be held parallel to the old position of A B C D, and just twice as far off from S. Let it be of such dimensions as just to
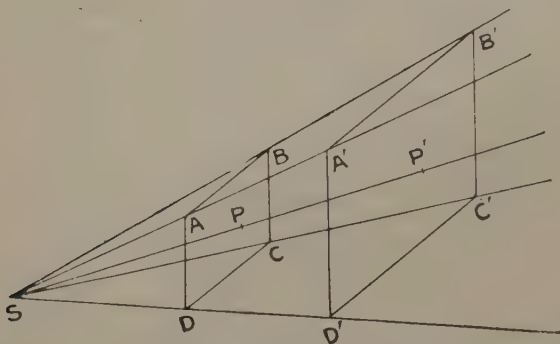


FIG. 3.

receive the illumination which fell on A B C D. It is clear that its linear dimensions must be double those of A B C D. Thus the mean illumination falling on A′ B′ C′ D′ per unit area is one-fourth of that which fell on A B C D. Or again, the mean illuminations per unit area at two points, P, P′, of the two surfaces in the same straight line with S are in the ratio of 4 to 1.

This reasoning may be easily generalized, and we see that, whatever be the distances of S from two surfaces, or two parts of the same surface, *the illuminations per unit area at two points*

*where the surfaces are equally inclined to their distances from* S, *are inversely proportional to the squares of those distances.*

This law of inverse squares may be verified experimentally. It is comparatively easy to judge whether two contiguous portions of the same surface, such as a screen of white, are equally illuminated, when they are illuminated from sources of the same quality, that is, giving light of the same colour, care being taken that the portion which is to be examined, that is illuminated by either source, is entirely screened from the other. This may be done by placing a rod so that the shadows formed of it on the screen by the two sources are contiguous; or by other methods shortly to be described. The portion in the shadow of the rod formed by one source is then illuminated only by the other source, and we have two adjoining portions of the screen illuminated, one by one source, and the other by the other. Now let five candles of the same sort be taken, and let one be used as one source, and four placed close together as the other source. It will be found that, to make the illuminations equal, the second source must be just twice as far off as the first. Thus we infer that the illumination on the screen due to a candle at a given distance is four times as great as if the candle is at twice that distance. The experiment may be varied by using different numbers of candles.

**Law of Cosines.**—We have hitherto considered how the illumination received at a point of a surface depends on its distance from the source when its inclination to that distance is unaltered. We shall now consider the effect of that inclination.

Consider a very narrow pencil from a distant point source falling on a surface, the pencil being so narrow that its
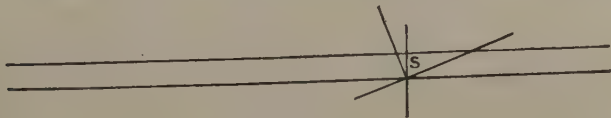


Fig. 4.

rays may be considered parallel. Let the surface at first be normal to these, and let $s$ be the infinitesimal portion of the area illuminated by the pencil. Now turn the surface so that it intersects the pencil at the same distance from the source, but so that its normal where the pencil meets it makes an angle, $i$, with the rays of the pencil. The area which now

receives the illumination of the pencil is $\dfrac{s}{\cos i}$; that is, the same amount of illumination is now received by a surface equal in area to the original one $\div \cos i$; or, the illumination received per unit area is equal to what it was in the former case $\times \cos i$. Thus we have the law that the illumination per unit area received by a surface is proportional to the cosine of the angle at which its normal is inclined to its distance from the source of light.

**Radiating Surface.**—Consider a surface radiating light, and which is equally bright all over, that is, emitting the same amount of illumination per unit area of surface at all points, such as a red-hot ball of metal; or, roughly speaking, the sun would be an example. Experiment shows that to a distant eye all parts of this surface appear equally bright. This means that two portions of the surface subtending equal solid angles, or having the same apparent size at the eye, send equal amounts of illumination to it, no matter what their inclinations may be.

Now consider two small portions of the radiating surface at A and C sending illumination to a distant eye along A B and C D. Let the normal at $s$ be along A B, and that at $s'$ inclined at an angle $i$ to C D. Let $s$ and $s'$ be of such sizes as to appear equally large to the eye. Then (as stated above) experiment shows that $s$ and $s'$ send equal amounts of
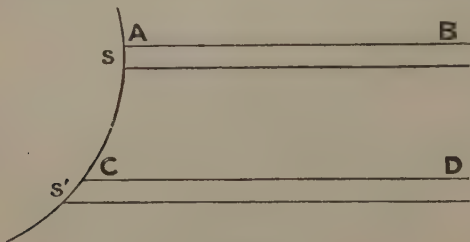


FIG. 5.

illumination to the eye. But the apparent size of $s'$ is the orthogonal section of the cone C D at C, that is, $s' \cos i$. Thus $s'$ emits in the direction C D the same amount of illumination as would be emitted normally by a portion of area $s' \cos i$. And it follows that *the illumination emitted in any direction by a given element of illuminating surface is proportional to the cosine of the angle that the normal to that element makes with that direction.*

**Solid Angle.**—It will be well to explain here fully what is meant by a solid angle. Suppose we have a sphere of radius unity. The solid angle which any portion of its area subtends at the centre is measured by the number of units of area in that portion. This is also called the solid angle of the cone which has its vertex at the centre and whose sides pass through the contour of the given portion of surface. Now imagine another sphere concentric with the first, and of radius $r$. Consider the portion of its surface cut out by the same cone. This subtends the same solid angle at the centre as the portion considered of the first sphere. And the portion of its area subtending this angle is $r^2 \times$ the corresponding portion of the other sphere. The solid angle may be measured by *area of sphere* $\div r^2$.
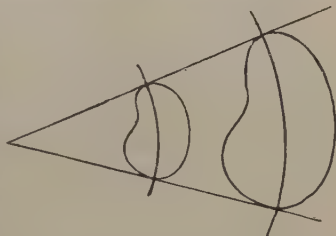
Fig. 6.

Any element of area $s$ subtends at a given point O, distant $r$ from $s$—$r$ being normal to $s$—a solid angle whose measure is $\frac{s}{r^2}$. If the normal to $s$ makes an angle, $i$, with $r$, to find the solid angle subtended by $s$ at O, we must consider what area at the same distance $r$ from O, and normal to $r$, would subtend the same solid angle at O. This is the orthogonal
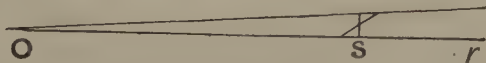
Fig. 7.

section at $s$ of the elementary cone, having O as vertex and $s$ for an oblique section. And the area of this orthogonal section is $s \cos i$. Thus, generally, the solid angle subtended at O by $s$, at a distance $r$ from O, and having its normal inclined at an angle $i$ to $r$, is $\frac{s \cos i}{r^2}$.

From the above two laws, namely, the law of inverse squares and the cosine law, it follows that the amount of illumination received at a point from any portion of a uniformly bright surface depends simply on the solid angle subtended by that portion at the point. For suppose an element of area $s$, at distance $r$ from a given point, and having its normal inclined

at an angle $i$ to $r$. The illumination which $s$ sends to the point is proportional to $\frac{s \cos i}{r^2}$; but this is the solid angle subtended by $s$ at the point.

## PHOTOMETRY.

It is a matter of considerable practical importance to be able to measure the amount of illumination emitted by a given source or to compare it with that emitted by a standard illuminator. The standard illuminator in general use is a sperm candle burning at the rate of 120 grains per hour. Other more convenient sources may be used, as, for instance, a gas flame with an aperture of definite size through which its light is emitted. Illuminators, it should be noticed, too, are not, as a rule, equally bright all round; it may be necessary to compare the light emitted by them in various directions with the standard light.

By the **illuminating power** or **intensity** of a source of light, we mean the ratio of the light emitted by it to that emitted by the standard source; or the ratio of the illuminations thrown by the two sources normally on surfaces of equal extent and at equal distances from the sources.

By the **intrinsic luminosity** of a surface, we mean the quantity of light, as compared with that emitted by a standard source, emitted per unit area of the surface normally, or emitted per unit of apparent area of the surface in any direction, since, as we have seen, the surface appears equally bright when viewed at any inclination.

To compare the intensities of two sources, the plan always adopted is to cause the sources to illuminate two surfaces, so arranged that they can readily be compared with each other, and to adjust the distances of the sources from the surfaces till the surfaces are equally illuminated, or appear equally bright—care being taken that each surface receives no other illumination than that coming from one of the sources. The distances $d$, $d'$ of the surfaces from the sources are then measured; and if I, I' are the intensities of the sources, since the surfaces are equally illuminated, or there is the same amount of illumination per unit area at each of them, we have—

$$\frac{\text{I}}{d^2} = \frac{\text{I}'}{d'^2}$$

Or—

$$\frac{I}{I'} = \frac{d^2}{d'^2}$$

Thus the illuminating powers are in the direct ratio of the squares of the distances. This operation is called *photometry*, and any apparatus by which it is carried out is a *photometer*. Various photometers have been employed for comparing intensities.

**Rumford's, or Shadow Photometer.**—The two lights to be compared are placed before a screen, say of white paper, and a rod is placed between them and the screen so as to cast two shadows on the screen. The lights and rod are adjusted so as to make the shadows contiguous and equally dark. The lights and the rod should be nearly in the same normal to the screen, so that the portions of surface examined should be illuminated normally. Now, the surface in the shadow formed by each light is illuminated only by the other; so that the illuminations produced by the two lights on the part of the screen examined are equal. If, then, the distances of the lights from the common boundary of the shadows where the illuminations appear equal are measured, the squares of these distances are in the ratio of the intensities.

**Bouguer's Photometer.**—The illuminated screen is made of ground glass, tissue paper, or some substance which will show the illumination behind. A blackened screen or diaphragm is placed normally to this, dividing it into two portions. The two lights are then adjusted close to the blackened diaphragm, so as to illuminate the two portions of the screen into which it is divided by the edge of the diaphragm, equally at their common boundary. By measuring the distances of the lights from this boundary, we can compare the intensities as before.

**Bunsen's Photometer.**—Bunsen adopted the plan of using a screen in which one portion, in the centre, is made more translucent than the rest by means of a spot of grease. The lights are placed on the two sides of the screen. Then, if the two sides are unequally illuminated, the grease spot will appear darker than the rest on the more illuminated side, and brighter on the other side. By arranging the lights so as to produce similar appearances on the two sides, we know that we have produced equal illuminations.

In Letheby's modification there are two mirrors, M, M', on the two sides of the screen S, so that the two sides may be

seen at the same time by the two eyes. The lights are at I, I';
and the screen with its mirrors can be moved to and fro along
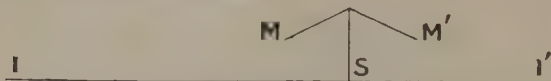a divided scale between them.



Fig. 8.

Of these photometers the last described gives the best
results for purposes of accuracy. Rumford's is a very con-
venient arrangement for forming a rough estimate of the
comparative values of two lights, as the presence of other
lights, or of stray light in the room, produces little effect.

In comparing the values of different lights, the question
of quality is of great importance, that is, the colours of the
lights, one light having more of blue, while another has more
of red in it. Two observers may form very different estimates
of the comparative values of two lights, simply because of the
different ways in which their eyes are affected by them, some
eyes being more sensitive to red light and others to blue.
Thus the light from an arc lamp is much bluer than that from
a gas flame; and so an observer having eyes very sensitive
to blue light may give a higher value of the intensity of the
arc lamp, when compared with a gas flame, than another
would. To get over this difficulty, the following plan is
frequently adopted: Two sets of observations are taken,
letting the light from the two sources pass in the two cases
through red glass and through green glass; and the mean of
the results so obtained is taken.

### EXAMPLES.

1. A luminous circular disc 1 foot in diameter is placed at 12 feet
distance from a screen and parallel to it; an opaque disc 2 feet in
diameter is placed symmetrically between them, and at a distance of 4
feet from the screen: find the diameter of the umbra and of the
penumbra.

2. A luminous circular disc 10 cms. in diameter is placed at 100 cms.
distance from a screen, and parallel to it: find the least distance from the
disc at which an opaque square 5 cms. in the side must be placed, parallel
to it and the screen, so that there shall be no complete shadow.

3. A uniformly bright surface is looked at through an aperture held
close to the eye, and so narrow that the border of the surface is never seen:
explain why the amount of light received by the eye is independent of the
distance of the surface.

4. In using Letheby's photometer to measure the candle-power of a
lamp, the standard light of 2 candle-power is set at a point which is found
to be 2·7 cms. on the negative side of the zero of the scale; and the lamp

is set at the reading 130 cms.; the screen receives equal illuminations when it is at 25·5 cms.: find the candle-power of the lamp.

5. Of two small, equally bright surfaces, one is a square and the other a circle whose radius is equal to the side of the square; they are set at distances in the ratio 1 : 2 from a point at which they are to produce equal illuminations: if the circle is at right angles to the line joining it with the point, find how the square must be placed.

# CHAPTER II.

## *REFLEXION. MIRRORS.*

A BEAM of light falling on a polished surface undergoes reflexion; that is, after striking the surface, it passes away from it in a definite direction, depending on its direction before meeting the surface. The polished reflecting surface is generally spoken of in optics as a **mirror**.

The manner in which the direction of light falling on a polished surface is modified is specified in the laws of reflexion, which we shall shortly state. These laws refer to a single ray of light falling on the mirror, and reflected off as a ray. They may be taken as experimentally proved for a very limited beam or pencil which approximates to a straight line. And, with regard to their application, for any finite pencil, by tracing the positions of the various reflected rays, we can determine the whole reflected pencil. Before stating the laws, it will be useful to explain the meanings of some terms.

The light falling on a reflecting surface is called incident light; and any ray of this light is called an **incident ray**. Any ray of the reflected light is called a **reflected ray**.

Suppose the normal to the mirror at the point where the incident strikes it and where the reflected ray leaves it to be drawn; then the plane containing the normal and the incident ray is called **the plane of incidence,** and the plane containing the normal and the reflected ray is called **the plane of reflexion.**

The angle between the incident ray and the normal is called the **angle of incidence**; the angle between the normal and the reflected ray is called the **angle of reflexion.**

The **Laws of Reflexion** are these—

I. *The incident ray, the normal ray, and the reflected ray are in one plane.*

II. *The incident and the reflected rays make equal angles with the normal.*

These laws may be approximately verified as follows : A circle is divided into degrees. Two tubes are set on it, each having a diaphragm wit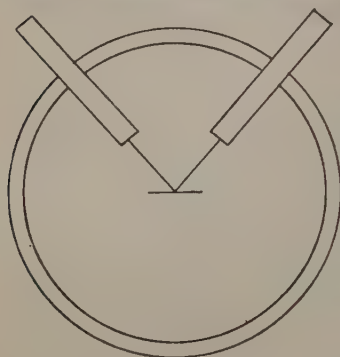h a small aperture at each end, and with the line of the apertures pointing to the centre of the circle and parallel to its plane, the tube being capable of rotating about an axis through the centre of the circle and at right angles to it. The lines of apertures of the two tubes move in the same plane parallel to the circle. A small mirror is set on the circle, at its centre, at right angles to the line joining the divisions 0° and 180°. Now, if light is allowed to fall through one tube on the mirror, the other tube may be set to receive the reflected light,—which proves the first law ; and when it is so set it will always be found to make the same angle with the line 0°–180°, which is normal to the mirror, as the first,—which is the second law.

FIG. 9.

This, as has been said, is only an approximate method of verifying the laws, since the experiments described do not admit of very great accuracy. What must be taken as the most rigorous proof is the consistency of the results got by assuming the laws as the basis of observations susceptible of a high degree of accuracy. Similar remarks apply to a great many, if not to most, physical laws ; the best proof of the laws being the uniform consistency of the results got by assuming their truth, the experiments adapted for proving them directly not being capable of great accuracy.

As we have said, the light by which we see an ordinary unpolished surface is light which has first fallen on it and then passes off again. This light is sometimes called *diffused*, or *irregularly reflected*. It is generally only a much smaller portion of the light that falls on the surface than regularly reflected light is. It does not follow the laws of reflexion, but passes off from the surface in all directions. There is no perfect reflector. Any surface will diffuse some of the light that falls on it. It is the diffused light only that renders the body visible ; regularly reflected light only showing images of other objects.

**Image of Point formed by Plane Mirror.**—Let A be
a luminous or visible point. Let us consider how it would be
seen by means of light reflected from a plane mirror. Take
any ray from A to the mirror, and suppose this to lie in the
plane of the paper : let it be A P. Let the normal from A to
the mirror be also in the plane of the paper. Let X Y denote
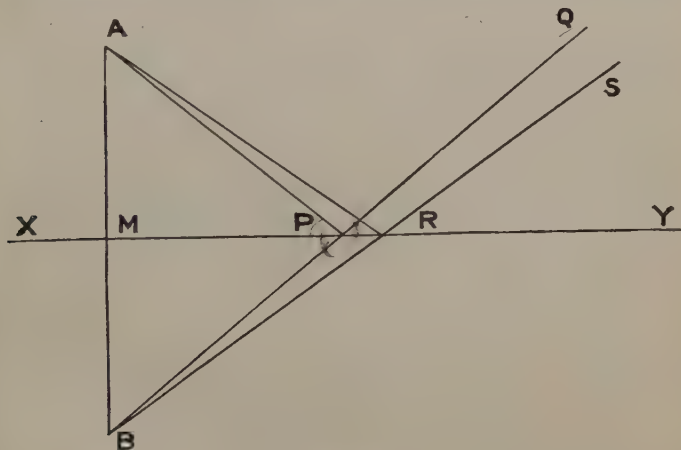the intersection of the paper with the mirror, which is thus at



FIG. 10.

right angles to the paper. The plane of the paper, being the
plane of incidence, will also be the plane of reflexion. Let
P Q be the reflected ray from the point P. A pencil of rays
such as P Q, R S, will proceed from the mirror, and would be
received by an eye if directed along Q P.

Now draw A M perpendicular to the mirror, and produce it
to B, making M B equal to A M. Join B P. Then in the triangles
A M P, B M P, the angle B P M = the angle A P M, and ∴ = the
angle Q P Y. And Q P is in the same plane with B P and X Y ;
∴ B P Q is a straight line. Thus the ray P Q, and similarly all
the reflected rays, would, if produced backwards, pass through
the point B. To an eye directed along Q P, therefore, the
reflected pencil of light which enters it produces the same
appearance as a visible point similar to A placed at B, the
mirror being removed.

**Images.**—The point B is called the image of A. It should
be noticed that the reflected light is just the same as if it had
really come from B. The same action is going on in the

pencil P Q as if A and the mirror were removed and a visible point were placed at B to emit this pencil. An object from whose various points pencils proceed will form an image consisting of the assemblage of images of its various points. If the pencils of light proceeding from the various points of an object undergo change of direction, so that they then proceed, or appear to proceed, from another assemblage of points, this new assemblage of points is called the image of the object. If the deviated rays actually pass through the points of the image, the image is said to be **real**. If the deviated rays only appear to pass through the points of the image, the image is said to be **virtual**. Thus the image of a point formed by reflexion in a plane mirror is virtual.

**Image of Object formed by Plane Mirror.**—This may be found by constructing the images of the various points of the object. It is such that the straight line joining any point of the object with the corresponding point of the image is bisected at right angles by the reflecting surface. The diagram
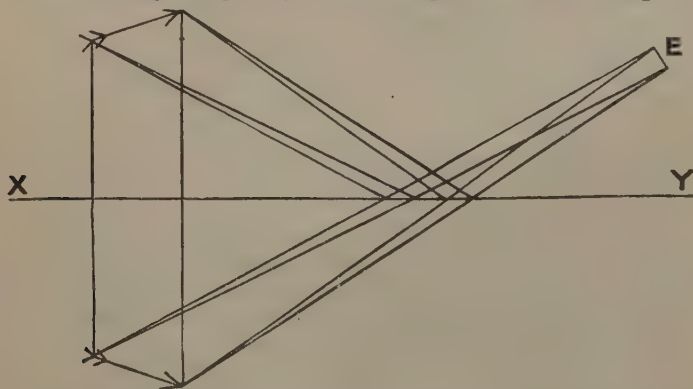


FIG. 11.

shows how the pencils of light from the various points of an object reach an eye situated at E, after reflexion at the mirror X Y, in the same manner as if they had come from the image of the object.

**Lateral Inversion.**—The points of the image have not in all respects the same relative positions as the points of the object. If, for example, the image formed in a plane mirror of a printed page be observed, then all the letters and words will be inverted, so that, the top and bottom keeping their places, the right-hand side of each letter becomes the left, and

*vice versâ*, and the words and lines will run from right to left instead of from left to right. If the image of the face be observed, the image of the right side is the left side of the image. This inversion which is produced in the image is called *lateral inversion.*

**Reflexion of Convergent Pencils by a Plane Mirror.**—Suppose P is the focus of the converging pencils, so that all the rays are, before reaching the mirror, proceeding
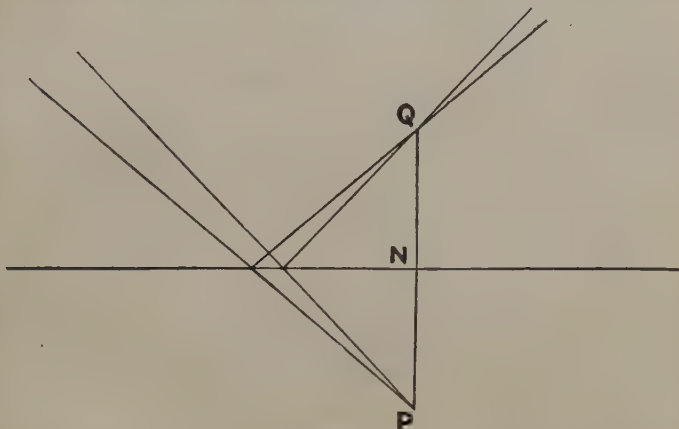


Fig. 12.

to the point P behind the mirror. Draw P N perpendicular to the surface, and take Q on the other side, so that Q N is equal to N P. Then we may show, just as for a diverging pencil, that all the rays after reflexion pass through Q. Thus Q is the focus of the reflected pencil; or Q is the image of P formed by the mirror. An eye in front of the mirror will see a *real* image at Q.

If a real image of an object is produced by any means, and a plane mirror is placed in the way of the pencils, so that the image cannot be formed where it would have been, a real image will be formed by the reflected rays. And this image is constructed from the other, in the same way as a virtual image in a plane mirror from the object, by drawing perpendiculars and producing them.

**Reflexion at Several Plane Surfaces.**—Pencils of light from a visible object may undergo reflexions at more surfaces than one before reaching the eye. To consider, first, the case of two reflexions; it must be remembered that the light, after

the first reflexion, is proceeding in just the same way as if it had come from an object coinciding with the image formed by the first mirror. To find the result of the second reflexion, therefore, we have merely to consider this image as an object, and so construct the image of it in the second mirror. This, again, may give rise to another image in the first mirror, or perhaps in a third mirror; and so on.

Suppose we have two mirrors only. For an object to be able to give an image in either mirror, it must be in front of the mirror, that is, so that perpendiculars drawn from it to the plane of the mirror meet it on the reflecting side. The same, too, is true for any one of the images. If there is formed in one mirror an image which is behind the other mirror, this image can give rise in the latter mirror to no new image. Rays of light proceeding from it (or, as if they came from it) could not strike the latter mirror.

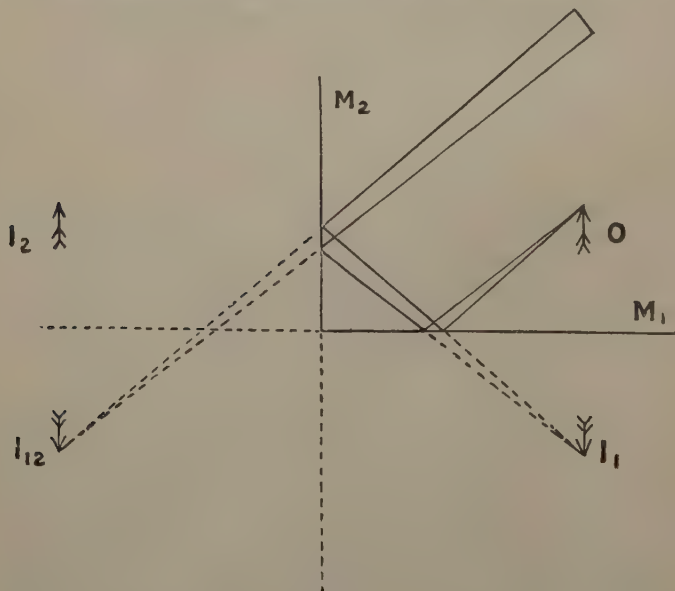*Two mirrors at right angles.*—Call the mirrors $M_1$, $M_2$.



Fig. 13.

Let O be the object. This will give rise to images $I_1$, $I_2$ in the two mirrors. $I_1$ will give in $M_2$ an image which may be

called $I_{12}$. This would be seen by an eye so situated that $M_2$ lies between it and all points of $I_{12}$. Again, $I_2$ will give in $M_1$ an image which we may call $I_{21}$, but which will coincide exactly with $I_{12}$, so that they may be called one image, $I_{21}$; that is to say, an image formed by reflexion first in $M_2$ and then in $M_1$ would be seen by an eye so situated that $M_1$ comes between it and all points of the image $I_{21}$ or $I_{12}$. An eye may be so situated that the image of O formed by two reflexions is partly $I_{12}$ and partly $I_{21}$. The intersecting edge of the mirrors would be seen to cross this image, and a part of it would be seen in each; but if the mirrors are properly adjusted, these two parts would fit exactly, wherever O may be. In this case three images are formed. A pencil of light by which $I_{12}$ is seen is shown in the diagram.

*Two mirrors inclined at* 60°.—The diagram shows the images that are formed. The image marked $I_{212 \atop 121}$ may be formed by reflexions in the mirrors either in the order $M_2$,
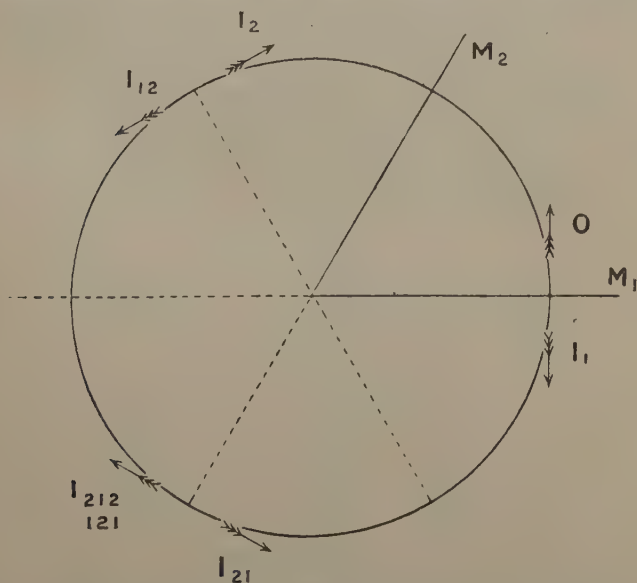


FIG. 14.

$M_1$, $M_2$, or in the order $M_1$, $M_2$, $M_1$, according to the position of the eye. In this case five images are formed.

It may be seen that, in general, the two cases considered being particular examples, if two mirrors are inclined at an angle which is the $n$th part of four right angles, $n$ being an integer, there will be $n - 1$ images.

It should be noticed, as the above diagram indicates, that the object and its images are arranged on the circumference of a circle with its centre at the intersection of the mirrors. The symmetry of the arrangement of the object and the images in pairs should also be noticed.

**The Kaleidoscope.**—This optical toy was invented by Sir David Brewster. It consists of three, or sometimes of two, equal strips of silvered glass, arranged at angles of 60°, and having their reflecting sides turned towards each other, and contained in a tube. At one end of the tube is a pair of transparent glass plates set across the axis of the tube, and containing pieces of coloured glass or other small objects. On looking through the other end of the tube, there is seen a symmetrical arrangement of similar patterns, consisting of the figure seen through the triangular aperture and its various reflexions. As the pieces in the end are moved about by turning the tube, these patterns continually change. The kaleidoscope is employed by designers.

In all the cases we have considered, the number of images formed is finite. The number may, however, be, theoretically, infinite, although the brightness of the images gradually becomes less and less.

*Two parallel mirrors.*—In this case there is an infinite
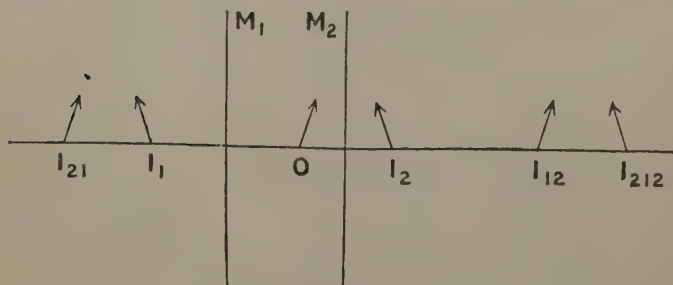


FIG. 15.

succession of images arranged, with the object, along a straight line perpendicular to the mirrors, as the diagram shows.

A pretty experiment may be performed with two mirrors and a lighted candle. Place the candle between the mirrors,

and hold these inclined at a small angle. A series of images of the candle will be seen arranged in a circle in pairs equally distant all the way, or with equal distances between each two successive images if the candle is equidistant from the mirrors. As the mirrors are brought nearer and nearer to parallelism, the circle enlarges, and more images appear; and when the mirrors are parallel, the circle becomes the straight line of the last case.

**Measurement of Angular Deflexion by Reflected Light.—Poggendorff's Method.—**To measure the angle through which a body is rotated about the vertical, a vertical mirror is attached to it. Let a horizontal ray of light fall on this mirror. Suppose it to make an angle, $\alpha$, with the normal. Then the reflected ray will make an angle, $2\alpha$, with the incident ray. If, now, the mirror is rotated through an angle, $\beta$, the reflected ray will make an angle, $2\alpha \pm 2\beta$, with the incident ray; that is, it will be deflected through an angle, $2\beta$. Thus if we measure the deflexion of the reflected light, that of the body to which the mirror is attached is half as much.

The deflexion of the reflected light may be measured in the following manner. The figure shows a plan of the arrange-
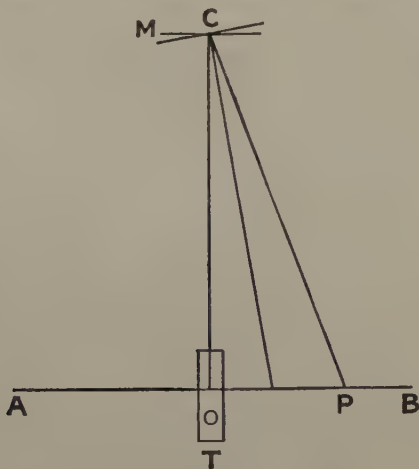


Fig. 16.

ment. A graduated scale, A B, is placed on about the same level as the mirror M, and parallel to it in its undeflected position. Let $d$ be the distance between them. A B is viewed

by reflexion in M by means of a telescope, T, placed just under A B. When M is undeflected, the reading of the scale at O, just above the telescope, should be seen at a marked position in the telescope, generally to coincide with a vertical wire fixed in the telescope. C O is perpendicular to A B. When the mirror is deflected through an angle, $\theta$, suppose the reading at P is seen in the telescope in the marked position. Then we have $\tan 2\theta = \dfrac{OP}{d}$.

## SPHERICAL MIRRORS.

A spherical mirror is a portion, generally very small, of a reflecting spherical surface. If the reflecting surface is on the concave side, towards the centre of the sphere, the mirror is called **concave**; if on the convex side, away from the centre of the sphere, the mirror is called **convex**. The centre of the sphere of which the mirror is a portion is called the **centre**, or **centre of curvature**, of the mirror. If the mirror has a circular boundary, its middle point is called its **pole**. And the straight line joining the centre to the pole is called the **principal axis**.

We shall investigate the relative positions of a small object and its image formed in a spherical mirror. The result is conveniently expressed in a formula. For the purposes of proving and applying this formula, it is necessary to have a convention with regard to the algebraical signs of the distances —which will all be measured from the pole of the mirror, and along its axis—occurring in the formula. The convention is that all distances measured *towards* the direction *from* which light is coming, or measured in the direction opposite to the incident light, are reckoned *positive*, and those measured in the *opposite direction* are reckoned *negative*. Thus with regard to the sign of the radius of the mirror: for the concave mirror, the light, striking the concave side, is travelling in the direction from the centre to the pole, so that the radius, measured from the pole to the centre, is positive; for the convex mirror the radius is seen, in a similar way to be negative.

Consider the formation of an image of a single visible point in a concave mirror. We shall suppose the point to be situated on what we have called the principal axis of the mirror; but is it clear that, wherever the point may be, if the straight line joining it to the centre meets the mirror, an image of it will be formed in the same manner with reference to this straight line.

Let the figure represent the section of the mirror by the plane of the paper, C being the centre, and A the pole. Let P be a visible point on the axis. Draw P R, a ray from P to the mirror. To find the position of the reflected ray from R, first it must lie in the plane with P R and C R, which is normal to the mirror at R; that is, it must lie in the plane of the paper, or
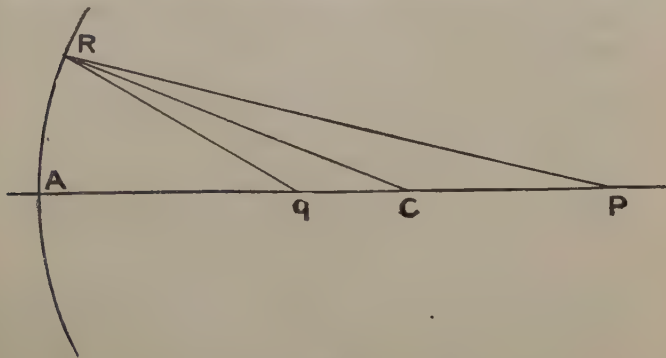


FIG. 17.

must intersect the principal axis (thus we see that all the reflected rays must meet the radius through P); next it must make the angle $q$ R C equal to the angle P R C. Now, all the rays from P do not, after reflexion, pass through the same point on the axis. But there is an ultimate limiting position of the point through which rays will pass if they are taken indefinitely close to the axis. This is the point at which a true image of P will appear to an eye looking along P C A, or looking just a little to the side, so that P is not in the way of the image; this image being formed by rays meeting the mirror indefinitely near to A. An eye would receive light which comes from P by reflexion at any other part of the mirror, but nowhere else would a true and distinct image be seen. This case differs from that of the plane mirror. A true image is seen by reflexion at any part of a plane mirror, and always in the same position. Notice that here a *true* image has been spoken of; this must be distinguished from a *real* image. A *true* image may be *real* or *virtual*.

Let Q be the ultimate limiting position of $q$, as the rays from P which, after reflexion, pass through $q$ become indefinitely close to the principal axis. Let the radius of the mirror = $r$. Let the distance of object from pole = $u$; and distance of image from pole = $v$. So that $u$ and $v$ in our

figure are both positive as well as $r$. We wish to find the relation between $u$, $v$, and $r$. From the figure we have, since R C bisects the angle P R $q$—

$$\frac{\mathrm{RP}}{\mathrm{R}q} = \frac{\mathrm{CP}}{q\mathrm{C}}.$$

In the limit, when R becomes indefinitely near to A, so that $q$ coincides with Q, this relation becomes—

$$\frac{u}{v} = \frac{u - r}{r - v};$$
$$\therefore u(r - v) = v(u - r),$$
$$r(u + v) = 2uv.$$

This may, again, be written, on dividing by $r\,u\,v$—

$$\frac{1}{u} + \frac{1}{v} = \frac{2}{r}.$$

This relation is quite general for spherical mirrors, concave or convex; and for all positions of object and image, provided due regard is had to the signs of $u$, $v$, $r$. We have taken here the simplest, and what is called the typical, case, in which all three quantities are positive. But if any other case is examined, the same formula will be found to hold. Let us, for example, examine the case of the formation of an image in a convex
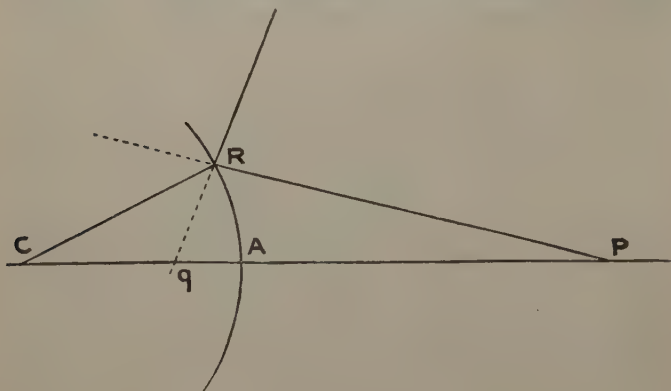


FIG. 18.

mirror. Let the visible point P (Fig. 18) on the principal axis send out rays, of which one, P R, after reflexion, appears to have come from the point $q$ on the axis. The limiting position of $q$,

when P R is indefinitely close to the axis, is a point, Q, the image of P. In the figure, C R bisects the exterior angle of the triangle P R $q$, got by producing P R. So that we have—

$$\frac{\text{RP}}{\text{R}q} = \frac{\text{CP}}{\text{C}q}.$$

In the limit this becomes—

$$\frac{\text{AP}}{\text{QA}} = \frac{\text{CP}}{\text{CQ}}.$$

This is a geometrical relation existing among the lengths of the lines, all four of which lengths are, here, to be taken as positive.

Let us now pass to the algebraical relation, and introduce the symbols $r, u, v$.

A P $= u$, $u$ being positive—

Q A $= v$ in absolute magnitude; but $v$ is here negative, so we must write—

$$\text{QA} = -v.$$
$$\text{CP} = \text{CA} + \text{AP} = -r + u.$$
$$\text{CQ} = \text{CA} - \text{QA} = -r + v.$$

Thus our relation becomes—

$$\frac{u}{-v} = \frac{-r + u}{-r + v}.$$

Changing the signs of both denominators, this gives, as before—

$$\frac{u}{v} = \frac{u - r}{r - v}.$$

Or—

$$\frac{1}{u} + \frac{1}{v} = \frac{2}{r}.$$

In this formula $r$ may be positive or negative, according as the mirror is concave or convex; $v$ may have either sign, its value depending on $r$ and $u$. But $u$ would, for such cases as we have considered, always be positive from the nature of the case; that is, whenever it denotes the distance from the mirror to a visible point.

There are, however, cases in which $u$ may be negative; and for these the formula holds too. Such a case is this: Suppose we have a convergent pencil of light, such, for instance, as would be produced by a concave mirror in the way we have examined. Now let a mirror be placed so as

to receive this pencil normally before it converges to a point, and reflect it so that it either converges to a point in front of the mirror after reflexion, or appears to diverge from a point behind the mirror. Then the point to which the pencil was converging takes the place of the object, and may in this case be called a **virtual object.** The other point to which the pencil converges after reflexion, or from which it appears to diverge, is the image of this object. The distance $u$, measured from the mirror to a point behind it, is negative.

The points P and Q are called **conjugate foci.** In the formula, $u$ and $v$ are interchangeable; they occur in the same way. If $u$ and $v$ have certain values satisfying the formula, then, if we give to $u$ the value of $v$, $v$ takes the value of $u$, Therefore if in any case an object at P gives an image at Q, an object at Q would give an image at P.

**Principal Focus.—Focal Length.**—There is one point on the principal axis of a mirror of great importance. Suppose that P goes off to an infinite distance ; or, what is the same thing, suppose a parallel pencil of light, indefinitely small and indefinitely close to the axis, to strike the mirror. Through what point do the rays pass, or appear to pass, after reflexion ?

The formula shows this. Making $u = \infty$, we get $v = \dfrac{r}{2}$. That is, the point is, for either mirror, midway between the pole and the centre. This point is called the **principal focus** of the mirror. We shall, in general, denote it by the letter F. The distance of F from the pole is called the **focal length** of the mirror. In both cases it is $\dfrac{r}{2}$.

For a concave mirror, the principal focus is the point through which the reflected rays from a parallel axial pencil actually pass ; or it is the real image of the point at infinity on the principal axis. It is called a **real principal focus.** The *focal length* is *positive.*

For a convex mirror, the principal focus is the point through which the reflected rays from a parallel axial pencil appear to pass ; or it is the virtual image of the point at infinity on the principal axis. It is called a **virtual principal focus.** The *focal length* is *negative.*

If in the formula we make $u = \dfrac{r}{2}$, we get $v = \infty$. From this, or from what has been said about the interchangeability of P and Q, it is seen that the principal focus has also this

property. An axial indefinitely narrow pencil diverging from F for a concave mirror, or converging to F for a convex mirror, will, after reflexion, be a parallel pencil along the axis.

Let us denote the focal length of a mirror by $f$, so that $f = \dfrac{r}{2}$; then the formula may be written—

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}.$$

To examine the various ways in which a concave or a convex mirror may act on an axial pencil, let us write the formula—

$$\frac{1}{v} = \frac{1}{f} - \frac{1}{u}.$$

The figures show the action in the various cases.

1. For a *concave* mirror—

(1) $u$ positive ; $v$ may be positive.



FIG. 19.

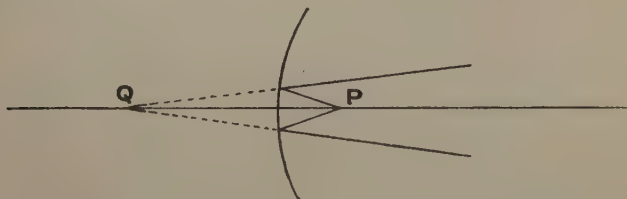(2) $u$ positive ; $v$ may be negative ; $v$ is then numerically greater than $u$.



FIG. 20.

(3) $u$ negative ; $v$ must be positive (Fig. 21).

In any case, it is seen that the pencil is more convergent (or less divergent) after reflexion. Thus the mirror is called a *converging mirror*.

2. For a *convex* mirror—

(1) $u$ positive ; $v$ must be negative (Fig. 22).

(2) $u$ negative ; $v$ may be positive ; $v$ is then numerically greater than $u$ (Fig. 23).

(3) $u$ negative ; $v$ may be negative (Fig. 24).
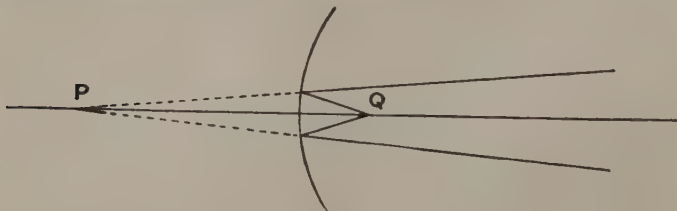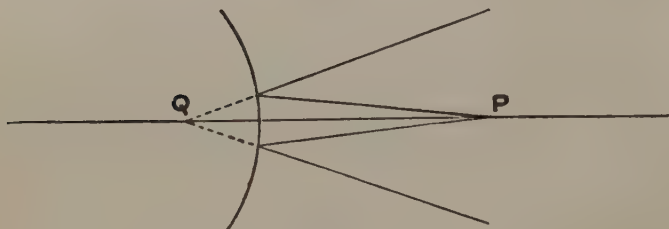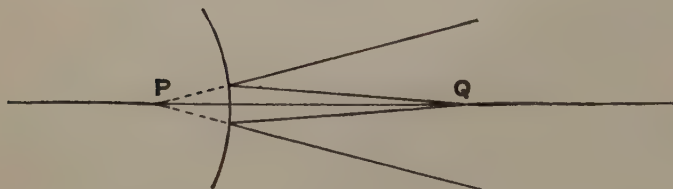


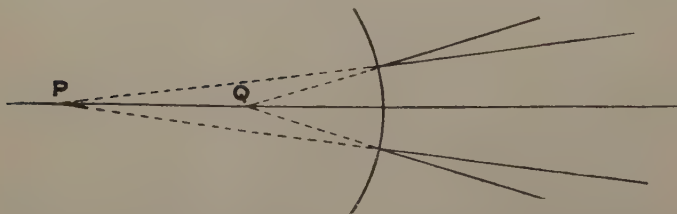FIG. 21.



FIG. 22.



FIG. 23.



FIG. 24.

In any case, it is seen that the pencil is more divergent (or less convergent) after reflexion. Thus the mirror is called a *diverging mirror*.

We shall now consider the variations in position of the image as the position of the object is changed. We shall make use of the formula—

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}.$$

1. *Concave mirror*—

When $u = \infty$, $v = f$; *i.e.* object at infinity gives image at F.

As $u$ decreases in value from $\infty$ to $f$, $\frac{1}{v}$ increases from 0 to $\frac{1}{f}$; $\therefore \frac{1}{v}$ decreases from $\frac{1}{f}$ to 0; $\therefore v$ increases from $f$ to $\infty$; *i.e.* as the object comes in from infinity to F, the image goes out from F to infinity.

When $u = r$, $v = r$; *i.e.* object and image meet and co-incide at the centre.

As $u$ decreases from $f$ to 0, $v$, which is now negative, increases algebraically from $-\infty$ to 0; *i.e.* as the object moves in from F to the mirror, the image, which is virtual, comes from an infinite distance behind to the mirror.

It should be noticed that the object gives a real or a virtual image according as it is beyond or within the principal focus.

2. *Convex mirror*—

When $u = \infty$, $v = f$; *i.e.* object at infinity gives image at F.

As $u$ decreases from $\infty$ to 0, $v$ is always negative, and increases algebraically from $f$ to 0; *i.e.* as the object comes in from infinity to the mirror, the image moves up from behind from the principal focus to the mirror.

It should be noticed that the image is always virtual.

We have only considered the cases in which the object is a real visible point; and the general conclusions drawn refer only to such cases. For instance, with a pencil of light converging to a point between the principal focus and pole of a convex mirror, the image formed would be real. Such cases as this may be considered just as the others by the help of the formula.

**Image of Small Object.**—Let A B be an object of dimensions indefinitely small compared with the radius of the mirror and the distance of the object from the mirror. The image of A B is the assemblage of the images of its various points. Every point of A B is at nearly the same distance from the mirror. This will be very approximately true indeed (or true to small quantities of the second order) if the small object is all in a plane perpendicular to the principal axis. $u$ having the same value for all points, it follows that $v$ will

have the same value, and the image is a small image at *a b*.
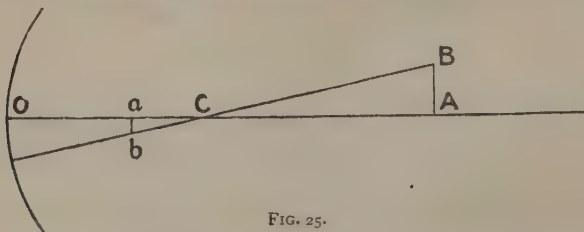When A B is all in a plane perpendicular to O C, *a b* is similar



FIG. 25.

to it, being in another plane perpendicular to O C, and every
straight line joining corresponding points passing through C.

**Graphic Construction of Images.**—We shall now
consider a convenient method of constructing graphically the
image, formed in a mirror, of a small object. These construc-
tions will depend on the following principle: A small pencil
near the axis gives by reflexion a pencil with a definite focus.
This focus we can find if we can construct geometrically any
two different rays of the pencil. For the focus is the point of
intersection of all the rays. Now, we shall see that there are
three rays of the incident pencil which, after reflexion, we can
follow up and draw. These are the following :—

(1) The ray which passes through the centre : this strikes
the surface normally, and is reflected through the centre.

(2) The ray which comes in parallel to the axis : this is
reflected through the principal focus.

(3) The ray which comes in through the principal focus :
this is reflected parallel to the axis.

Of course, when the mirror is convex, and the centre and
principal focus are behind it, we must consider rays whose
*directions* are through these points.

We shall now consider the three different general cases
that may occur of the formation of an image of a small object
in a spherical mirror, giving the graphic construction for each
case, the object being real.

All the rays drawn in these constructions are supposed to
be indefinitely close to the axis. Therefore, to represent them
in a figure of finite dimensions, we must suppose this figure
to be very greatly spread out in the direction at right angles
to the axis, but not in the direction of the axis. The trace of
the mirror on the plane of the paper will thus be a straight line
at right angles to the axis.

I. *Concave Mirror: Object beyond principal focus.*—Take A B, a small object, in a plane perpendicular to the axis, with the point A on the axis. The rays B R, B S, B T, from the point B of the object, are respectively through the centre, parallel to the axis, and through the focus; and they are reflected through
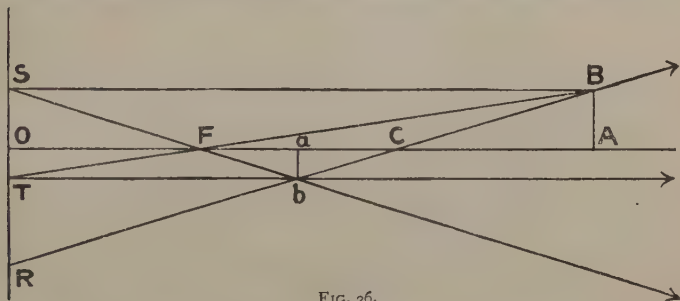


FIG. 26.

the centre, through the focus, and parallel to the axis. They give by their intersection the point *b*, the image of B. *b* is seen by means of a pencil diverging from it, of which there are three rays, these rays being indicated by arrow-heads. The position and magnitude of the whole image *a b* will be found by drawing *b a* perpendicular to the axis.

A B and the corresponding image may be drawn in other positions; for instance, if A B comes between the mirror and C, *a b* will be beyond C. But as long as A B is beyond F, the general characteristics of *a b* remain the same.

The image is *real* and *inverted*.

II. *Concave Mirror: Object between mirror and principal*



FIG. 27.

*focus.*—The same three rays as before, and indicated by the same letters, are here drawn. The pencil they give has here a

virtual focus, *b*. The entire image *a b* is constructed as before.
Notice that the action on the right-hand side of the points
R, S, T is physically the same, and produces the same effect
on the eye, as if it came from a real visible point *b*.
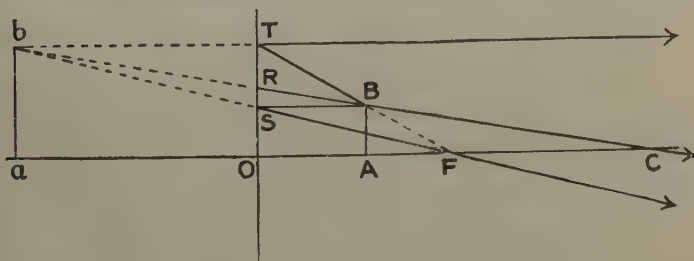
The image is *virtual* and *erect*.

III. *Convex Mirror.*—The construction follows the same
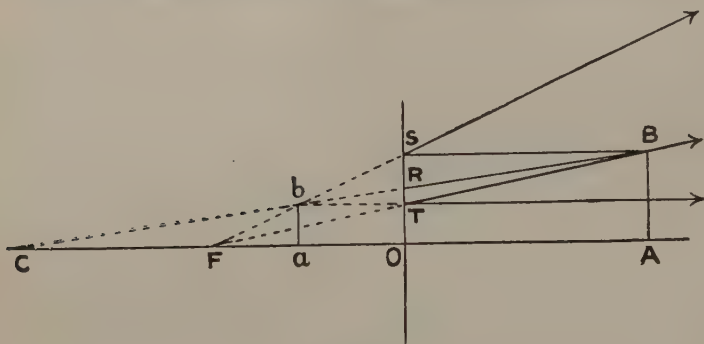


FIG. 28.

lines as before. However A B is moved about in front of the
mirror, the image will have the same general characteristics.

The image is *virtual* and *erect*.

In any case the image is necessarily erect if it and the
object are on the same side of C, and inverted if they are on
opposite sides; for the rays joining corresponding points of
object and image meet in C.

The image is real if on the same side of the mirror as the
object, for then the rays really pass through the image; it is
virtual if on the opposite side.

By referring to all the cases, it will be seen that the image
is real when inverted, virtual when erect.

**Magnification.**—When an image of a small object is
formed in a mirror, the ratio of any linear dimension of the
image to the corresponding dimension of the object—both
being measured at right angles to the axis on which object and
image are situated—is called the **magnification** produced by
the mirror.

In all the three figures that have just been given, the
magnification will be the fraction $\dfrac{ab}{AB}$. Important expressions
for the magnification may be deduced by reference to these

figures. What follows is applicable to any of these figures, and all three of the figures should be referred to at each step of the work. Let us denote the magnification by $m$. We have—

$$m = \frac{ab}{AB}.$$

From the similar triangles $abc$, ABC, we get—

$$m = \frac{ab}{AB} = \frac{Ca}{CA};$$

$$\textit{i.e. } m = \frac{\text{distance of image from centre}}{\text{distance of object from centre}} \quad . \quad (1)$$

This expression, which in symbols is $\frac{u-r}{r-v}$, is also equal to $\frac{u}{v}$ (see p. 25); so that we have—

$$m = \frac{\text{distance of image from mirror}}{\text{distance of object from mirror}}. \quad . \quad (2)$$

Approximately, O S is a straight line at right angles to O C, and equal to A B. Thus from the similar triangles $ab$F, OSF, we get—

$$m = \frac{ab}{AB} = \frac{ab}{OS} = \frac{a\text{F}}{O\text{F}};$$

$$\textit{i.e. } m = \frac{\text{distance of image from principal focus}}{\text{focal length}} \quad . \quad (3)$$

Approximately, O T is a straight line at right angles to O C, and equal to $ab$. Thus from the similar triangles OTF, ABF, we get—

$$m = \frac{ab}{AB} = \frac{OT}{AB} = \frac{OF}{AF};$$

$$\textit{i.e. } m = \frac{\text{focal length}}{\text{distance of object from principal focus}} \quad . \quad (4)$$

The numerical values, merely, of the quantities expressed in these fractions for $m$ are to be understood. Thus for a convex mirror O F is not the focal length, but is − focal length.

These four values of $m$ may be expressed in symbols as follows: The magnitudes whose ratios we shall write down are all so taken that they are positive in the first figure; but their numerical magnitudes will give the magnification in any case.

D

$$m = \frac{u - r}{r - v} = \frac{u}{v} = \frac{v - f}{f} = \frac{f}{u - f}.$$

Of the expressions given, perhaps number 4, which involves the position of the object and not that of the image, is the most important.

Some general inferences may be drawn with regard to the magnification produced in the various cases. If the magnification is greater than 1, so that the image, measured as specified, is greater than the object, the image is said to be *magnified ;* if the magnification is less than 1, the image is said to be *diminished.*

*Concave Mirror.*—When the object is beyond the centre, it is at a distance from F greater than $f$, so that from the value (4) of $m$ the image is *diminished ;* when the object is at C, the image is equal to it ; when the object is between the mirror and the centre, the image is *magnified.*

*Convex Mirror.*—The object is always at a greater distance from F than $f$; so that from the value (4) of $m$ the image is always *diminished.*

We shall now enumerate, for reference, the characteristics of the images formed by either mirror, and for all positions of the object, with regard to the following four particulars :—

(1) Position ;
(2) Whether real or virtual ;
(3) Whether erect or inverted ;
(4) Size.

*Concave Mirror.*—Object beyond C ; image between F and C, real, inverted, diminished.

Object at C ; image at C, real, inverted, equal to object.

Object between F and C ; image beyond C, real, inverted, magnified.

Object between O and F ; image behind mirror, virtual, erect, magnified.

*Convex Mirror.*—Object in front of mirror ; image behind mirror, between F and O, virtual, erect, diminished.

**Spherical Aberration.**—The rays of light in a broad pencil will not, after reflexion, give a pencil with a definite focus. An indefinitely narrow axial pencil gives, as we have seen, after reflexion, a pencil with a definite focus. But as rays are taken more and more inclined to the axis, they will, after reflexion, meet the axis farther and farther away from this focus. The figure shows how the rays from P, as they are taken at greater inclinations to the axis, meet the axis after reflexion at points farther from Q. This *aberration* of the

reflected rays may be considered as due to the form of the mirror, as it would be possible to construct a mirror which
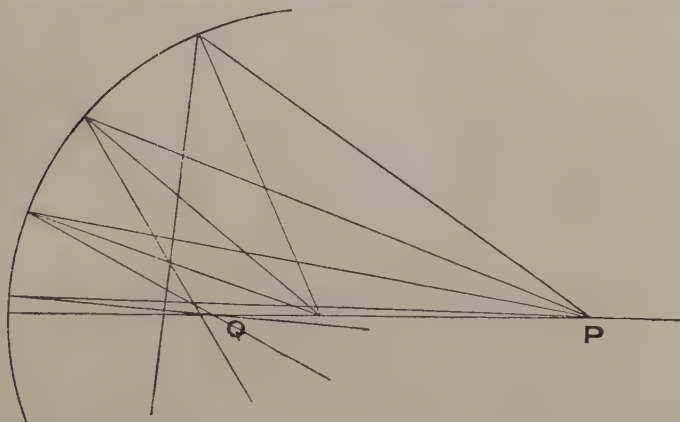


FIG. 29.

would bring all the rays to one focus. It is called **spherical aberration.**

We shall see now that mirrors may be made which, for particular broad pencils, do not produce any aberration.

**Parabolic Mirror.**—This is of the form of a paraboloid of revolution (got by rotating a parabola about its axis). If F
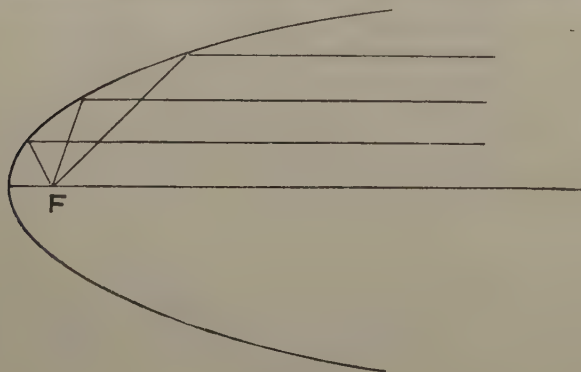


FIG. 30.

is the focus of the paraboloid, that is the principal focus of the mirror ; the normal at any point is equally inclined to the axis

and to the straight line joining that point to F. Thus any ray coming in parallel to the axis will be reflected to F. F is the focus for all rays parallel to the axis.

Again, all the rays coming from a luminous or visible point at F would be reflected parallel to the axis. Thus a parabolic mirror is frequently used for throwing a strong beam of light in a definite direction. A lamp is placed at the focus, and the axis of the mirror turned in the required direction.

**Ellipsoidal Mirror.**—This is of the form of an ellipsoid of revolution (got by rotating an ellipse about its major axis). If F, F' are the foci of the figure the normal at any point
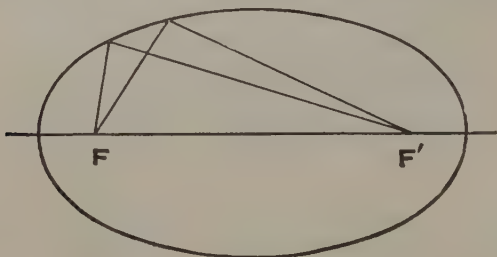


FIG. 31.

makes equal angles with the straight line joining the point to F and F'. Thus any ray from F is reflected to F'. Either of the points F and F' is the focus for all rays coming from the other.

**Determination of Focal Length.**—The focal length of a spherical mirror may be found by many methods.

If the radius of curvature is found in any manner, the focal length, which is half of the radius, is known.

By measuring corresponding distances from the mirror of a small object and its image along the line joining them, and substituting in the formula—

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

the value of the focal length of a concave mirror can be found.

If we get a small object into such a position that its image in a concave mirror coincides with itself, then the object and image are both at the centre. Thus we can determine the radius, and so the focal length.

The focal length of a convex mirror may be found by taking a pencil of converging light, and noticing the position

of its focus, then placing the mirror so that the light falls on it normally and is reflected as a convergent pencil; that is, the original focus must be between the mirror and its principal focus. Now measure the distance of the focus of the reflected pencil from the mirror, and calculate $f$ from the formula.

The position of the image formed in any of these methods may be found in either of the following ways :—

(1) Let the image be formed as distinctly as possible on a screen, moving the screen about till the best position is found. The screen is then in the position of the true image, and its distance from the mirror must be measured.

(2) Place a pointer or a stretched wire at about the place where the image is formed. Look at the pointer and the image together, and move the head from side to side; then if they appear to move with reference to each other, they are not in the same plane, and the pointer must be moved till no such motion is seen. It is then in coincidence with the image, and its distance from the mirror must be measured.

**The Optical Bench or Bank.**—This is an apparatus for making optical measurements such as those we have just described. It consists of a long base fitted with a graduated scale, and along which may be slid, in a groove or otherwise, stands carrying various supports adapted for carrying lights, screens, mirrors, and other optical apparatus. Each of these stands has a mark, which, by moving over the divided scale, serves to indicate the distance through which the stand may be moved. The supports should all be capable of some lateral and vertical adjustment on their stands, so that such points as may be desired of the objects that they carry may be brought into the same straight line, parallel to the bench.

A convenient method of measuring the distance between two objects, such as screen and mirror, on the bench, is to have a straight pointer of known length carried on one of the stands, and set parallel to the bench, and so that its points are in the line with the objects whose distance is required. Move the stand carrying the pointer till its points in turn touch the objects, taking the readings of the stand. The distance required is the difference of these readings plus the length of the pointer.

1. Show geometrically that with the help of two plane mirrors an image of a given straight line can in general be formed to coincide with any other straight line of the same length in a plane with the given one;

and that there is an infinite number of pairs of positions of the two mirrors for producing the required result.

2. Two plane mirrors make an angle, $\beta$, with each other. A ray of light travelling towards their intersection, and in a plane at right angles to it, makes an angle, $\alpha$, with the mirror on which it is first incident. Show that the condition that the direction of the ray may be completely reversed by $n$ reflexions at the mirrors—that is, that it may travel parallel to its old direction, but in the opposite sense—is—

$$2\alpha + (n - 1)\beta = \pi$$

or—

$$n\beta = \pi$$

according as $n$ is odd or even.

3. Light coming from a point 16 cms. in front of a concave mirror falls normally on its surface, and is then brought to a focus at 5·2 cms. from the mirror : find the focal length of the mirror.

4. A concave mirror of 5 feet focal length is used to form an image on a screen which is 20 feet from it : find how far from the mirror the object must be placed.

5. A concave mirror is used to bring a pencil of light to a focus at a point, O. A convex mirror is then interposed to intercept the light normally, so that O is 4 ins. behind its surface ; and the reflected pencil then comes to a focus at a point which is $6\frac{3}{4}$ ins. in front of this mirror : what is the focal length of the convex mirror?

6. The letter L is drawn on a sheet of paper and held upright before a concave mirror : describe, with drawings, the appearance that will be presented for all positions of the object.

7. If in the last question a convex mirror is used, describe the appearance that will be presented.

## CHAPTER III.

### *REFRACTION. LENSES.*

WHEN light passes out of one transparent medium into another it does not, as a rule, continue its path in straight lines in passing across the boundary surface of the media. If the light strikes the surface normally, it will continue to travel normally through it ; that is, in this case it will suffer no deviation. If the light strikes the surface obliquely, a part of it will, for almost any two media, be deviated at the surface, so as to travel in the second medium, if it is homogeneous, in straight lines, but in straight lines which are not the productions of those which formed the paths of the rays in the first medium. This action on light is called **refraction**. Another part of the light will be reflected at the surface, in accordance with the ordinary laws of reflexion.

This can be illustrated by many simple experiments. Observe

a small object under the surface of water, looking obliquely
to the surface.  Let a straight stick be held with one end near
the eye, and so as to touch the object with the other end.  It
is found that one must look, not along the stick, but above it ;
for the light from the
object, which reaches
the eye, does not
travel straight to the
eye in the direction
of the stick, but pen-
cils which leave the
object in a direction
nearer to the vertical
than the stick is, on
reaching the surface
are bent away further
from the vertical, and
so reach the eye.
The figure shows the
path of the rays
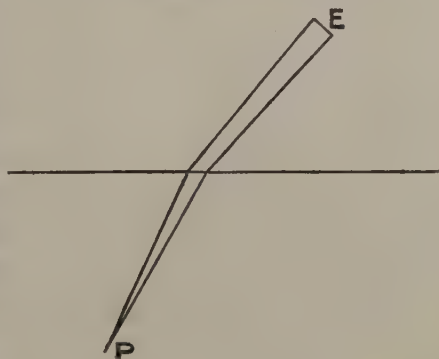


FIG. 32.

which leave the visible point P, and are thus bent at the
surface to reach the eye at E.

If a coin be placed in the bottom of a basin, and the eye
held so that the coin is just hidden by the rim, then, on pouring
water into the basin,
the coin will come
into view again.  For
the pencils by which
the coin would have
been seen at first,
before pouring in the
water, must have gone
straight to the eye ;
but after the water
is poured in light
reaches the eye from
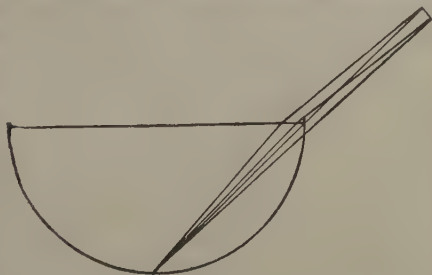


FIG. 33.

the coin which first goes more upwards, and at the surface is
bent away from the vertical ; thus getting round the rim of the
basin.  The accompanying figure shows the passage of the rays
in this case.

The appearance of a stick held obliquely, and partly in
the water, is another example.  The stick appears to be bent
upwards at the surface, and to be shorter than it really is.
For the pencils of light which reach the eye from it seem to

have come from points higher up than those from which they really come.

We shall now consider the precise manner in which light is refracted at the boundary surface of two media. We shall see that refraction depends on the nature of the media, being different in this respect from reflexion.

A ray of light falling on the surface is called, as in the case of reflexion, an **incident ray**; and the plane containing this ray and the normal to the surface at the point of incidence is called **the plane of incidence**, and the angle between the incident ray and the normal **the angle of incidence**. A ray going on into the second medium from the point of incidence is called a **refracted ray**; and the plane containing this ray and the normal is called **the plane of refraction**, and the angle between the refracted ray and the normal **the angle of refraction**.

The **Laws of Refraction** are these—

I. *The incident ray, the normal, and the refracted ray are in the same plane.*

II. *For the same two media, the sine of the angle of incidence bears a constant ratio to the sine of the angle of refraction.*

The second law is generally known as **the law of sines**; it was first given by Snell, a Dutchman, but is frequently attributed to Descartes.

We shall, in general, have to consider the case of light refracted from air into a given medium, or *vice versâ.*

When light is refracted from a vacuum into a given medium, the constant ratio of the sines is called the **refractive index**, or **index of refraction**, of the medium. This quantity is generally denoted by the symbol $\mu$. We shall see that the refractive index of a given medium differs for different colours; but, at present, and unless the contrary is specified, we shall suppose all the light considered to be of the same colour, so that $\mu$ is constant for a given medium.

When light is refracted from air into the medium, the ratio of the sines is practically the same as when it comes from a vacuum; that is, the air has practically no influence. If, then, $i$ and $r$ denote the angles of incidence and refraction from air or from a vacuum into a given medium, we have the relation—

$$\sin i = \mu \sin r.$$

The apparatus shown in the figure (Fig. 34) may be used to demonstrate the laws of refraction for liquids. There are two tubes, G H, L K, moving on a vertical circle, and always

directed to its centre, and provided with apertures as for the laws of reflexion. (Or the tubes may be provided with lenses, as the figure shows.) Instead of using the vertical circle the tubes may be prolonged with arms, in which are small holes at P, P′, these being equidistant from the centre of rotation. C O is a horizontal scale, which may be moved vertically, and the zero of which is vertically below the axis through the centre about which the tubes turn. C D E F is a vessel to contain the liquid, turning with the tube G H, and having a bottom, D E,
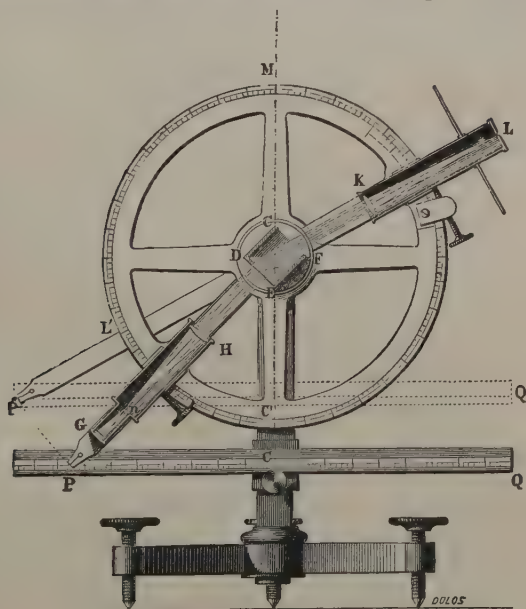


Fig. 34.

of plate glass set at right angles to G H, so that light crossing D E normally is undeviated. The surface of the liquid is adjusted to be level with the centre of the circle. K L is set in any position, and the reading through P′ taken. G H can be set to receive the light that passes through K L, and is refracted at the surface of the liquid, and, when it is so set, the reading through P is taken after suitably adjusting the scale. The ratio of C′ P′ to C P, which is the same as the ratio of the sines of the angles of incidence and refraction at the surface of the liquid, should be found constant.

Fig. 35 shows how the apparatus may be used to verify the law for solids. A B C is a half-cylinder of glass, or other transparent substance, set with the plane face A C horizontal, and the curved sides at right angles to the vertical circle. D E F K is a block of the same substance, with the face E F
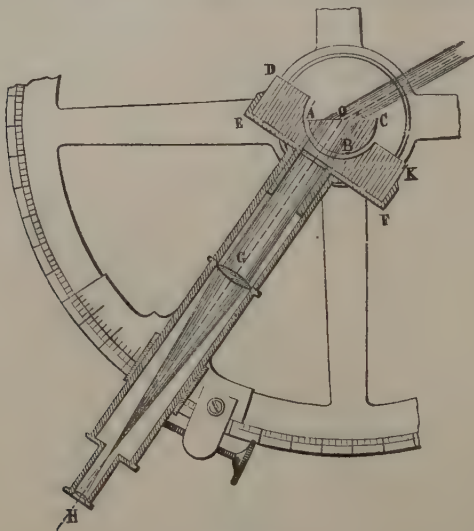


FIG. 35.

set at right angles to the tube G H, and having a cylindrical hollow into which A B C just fits. With these arrangements it is clear that the apparatus may be used in just the same way as before.

The laws of refraction, as those of reflexion, do not depend for their demonstration on such direct proofs as these, but on the consistency of the results obtained by assuming their truth as the basis of measurements that can be made with great accuracy.

When light is refracted from a medium $a$ into a medium $b$, the constant ratio of the sines of the angles of incidence and refraction is called the **relative index of refraction from** $a$ **to** $b$. This we may denote by $_a\mu_b$. There are some important relations existing between relative refractive indices, which we shall now consider.

It is known by experiment that the path of a ray of light is reversible. That is, if a ray along a given line in one

medium is refracted along a second line in another medium, then, when a ray passes along this latter line in the second medium, but in the opposite sense, it will emerge along the first line in the first medium. Thus if $i$ and $r$ are corresponding angles of incidence and refraction from medium $a$ to medium $b$, $r$ and $i$ will be corresponding angles of incidence and refraction from $b$ to $a$.

Since $_a\mu_b = \dfrac{\sin i}{\sin r}$, and $_b\mu_a = \dfrac{\sin r}{\sin i}$, it follows that $_b\mu_a = \dfrac{1}{_a\mu_b}$.

The following is an experiment in support of the above statement. If a plate of glass, that is, a portion with plane parallel faces, be held between the eye and a distant object, the direction of the rays falling on the glass and of those leaving it will be the same, whatever the obliquity to those rays at which the glass is held. The distant object will appear to be slightly displaced by an amount depending on the thickness of the glass and its obliquity. But the thinner the glass taken, the less will this displacement be, and the glass may be taken so thin as to make the displacement imperceptible to the eye. The figure shows the passage of the ray for this case. $i$ and $r$ are the angles of incidence and refraction at the first surface, at the point A. $r$ is thus also the



Fig. 36.

value of the angle of incidence, internally, at B, since the normals at A and B are parallel; and since the rays P A and B Q are parallel, the angle of refraction at B must be equal to $i$.
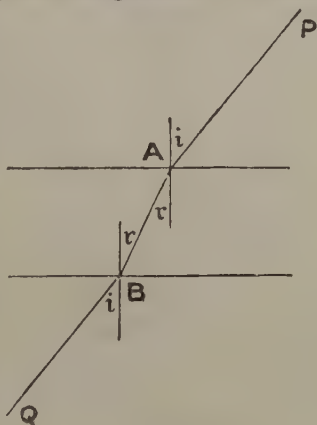
Suppose that a ray of light passes from a medium, $a$, through two plates of $b$ and $c$, having a common face, and then emerges into $a$. It is found that the rays P A and C Q in the two portions of $a$ are parallel. Now, let the angles of incidence and refraction be as shown in the figure. Then we have—

$$_a\mu_b = \frac{\sin i}{\sin r}; \quad _b\mu_c = \frac{\sin r}{\sin r'}; \quad _c\mu_a = \frac{\sin r'}{\sin i}.$$

Thus—

$$_a\mu_b \; _b\mu_c \; _c\mu_a = 1.$$

Or—

$$_a\mu_c = {}_a\mu_b \; {}_b\mu_c.$$

In the same way, experiment would show that any number of plates placed as indicated would not deviate a ray of light ; so that we should have general relations such as the above

existing among any number of relative refractive indices.

We may write the general relation for a number of media, $a$, $b$, $c$, . . . $k$, $l$—

$$_a\mu_b \; {}_b\mu_c \cdots {}_k\mu_l \; {}_l\mu_a = \text{I}.$$

As a particular application, suppose we have determined the refractive indices from air into glass and into water ; call these $\mu$, $\mu'$. Then the refractive index from glass into water is $\dfrac{\mu'}{\mu}$.
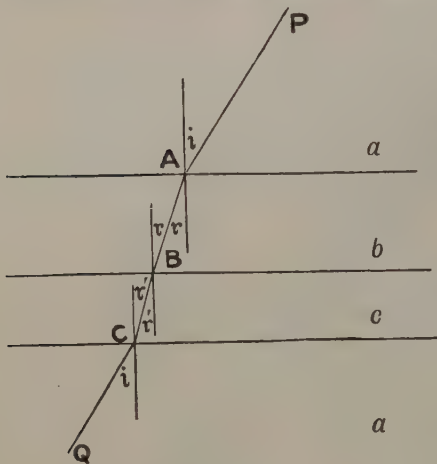
FIG. 37.

Let one medium be a vacuum ; denote it by $v$. Then we have—

$$_v\mu_a \; {}_a\mu_b \; {}_b\mu_v = \text{I}.$$

This we may write—

$$_v\mu_b = {}_v\mu_a \; {}_a\mu_b.$$

Thus *the absolute index of refraction of b is found by multiplying the relative index of refraction from a to b by the absolute index of refraction of a.*

In this case let $a$ be air, and $b$ some transparent medium, such as water or glass. We have said that we may consider $_a\mu_b$ and $_v\mu_b$ to be practically the same. This is because the correcting factor $_v\mu_a$, the index of refraction from vacuum to air, or *absolute index of refraction of air*, is found by experiment to be nearly equal to unity—about 1·0003.

In general, when light passes from a rarer to a denser medium, as from air to water, the rays are bent towards the normal—the refractive index is greater than unity. When the light passes from the denser to the rarer medium, the

light is bent away from the normal towards the surface of separation.

Suppose light to pass from a denser to a rarer medium (as when it comes up to the surface of still water and emerges into the air). Let $\mu$ be the index of refraction, which in this case is less than unity. For any angle of incidence, $i$ (measured inside the denser medium), we calcuicate the angle of refraction, $r$, by the relation $\sin r = \dfrac{\sin i}{\mu}$. Thus $r$ is always greater than $i$ in this case.

Fig. 38 shows how a series of rays, O A, O B, O C, O D, coming from a point O, are refracted at the surface. Any value of $i$ will give a possible value of $r$ as long as it gives $\sin r$ less than 1; that is, as long as $\sin i$ is less than $\mu$. Thus for all angles of incidence, from zero up to a certain value, for
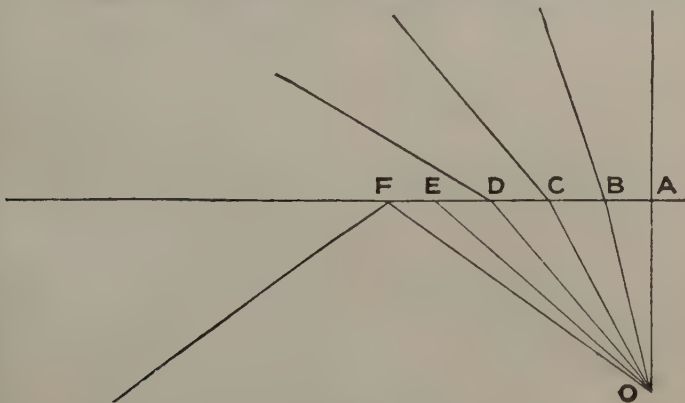


FIG. 38.

which $\sin i = \mu$, some of the light which reaches the surface is refracted into the second medium. Some will also be reflected back into the first medium. As $i$ increases, the refracted rays get closer and closer to the surface. For the ray O E making an angle with the normal whose sine is $\mu$, light emerges glancing along the surface. The question arises— What happens when the incident rays are beyond O E, as O F, making greater angles with the normal? In this case none of the light is refracted, but all which falls on the surface is reflected back into the first medium. This phenomenon is called **total internal reflexion**.

Suppose the second medium to be air. Then the limiting

ray O E makes with the normal an angle whose sine is equal to the index of refraction of the first medium. This quantity is denoted by $\frac{1}{\mu}$ above, since $\mu$ was taken to be the index of refraction from the first medium to the second. This angle is called the **critical angle** for the medium. If $\theta$ is the critical angle and $\mu$ the refractive index of a medium, we have—

$$\sin \theta = \frac{1}{\mu}.$$

The direction of a ray refracted from one medium to another may be found in practice by the following graphic method, when the index of refraction is known: Let X X denote the surface of separation of the two media. And let P O be a ray in the first medium incident on the second at the point O. With O as centre describe two circles with radii O A, O B in the ratio $1 : \mu$. Let OP cut the first of these in S; and draw
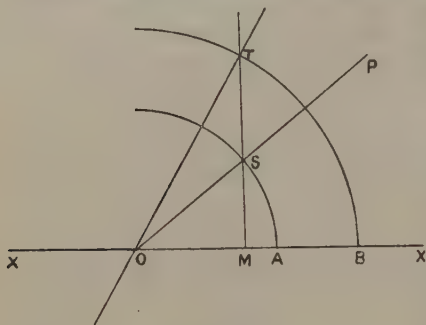


FIG. 39.

T S M perpendicular to X X, meeting the second circle in T. Then T O produced is the path of the ray in the second medium. For—

$$\frac{\sin \text{OSM}}{\sin \text{OTM}} = \frac{\text{OT}}{\text{OS}} = \mu.$$

And O S M is equal to the angle of incidence at O. Therefore O T M is equal to the angle of refraction.

If in this construction $\mu$ is $< 1$, O B is $<$ O A. In this case S M may or may not meet the second circle. If it does not, there is no refraction, and we have total internal reflexion. The limiting case is where S M touches the second circle at M. Then we have for the sine of the angle of incidence, which is now the critical angle, $\frac{\text{OB}}{\text{OS}}$; *i.e.* $\mu$.

To illustrate total internal reflexion: if the eye is held under the level of the surface of water in a glass, much more light is seen reflected from the under surface than if it be looked

at from above. If a rod is placed in the glass so as to be partly immersed, the image of it formed by reflexion in the under surface is as clear as that seen directly through the water.

In looking from above at an object under the surface of water, the object may be made to appear as close to the surface of the water as we please by holding the eye near enough to the surface. As the eye approaches indefinitely near to the surface, the pencils by which the object is seen meet the under surface at angles which approach to the critical angle, and emerge more and more nearly along the surface.

The rays which come from the outside to an eye under the surface of the water, as at O, must all make in the water angles with the vertical less than the critical angle. Thus this eye will see all objects above the surface as if they were contained in a cone with the eye at the vertex, and of semi-vertical angle equal to the critical angle. All these objects will appear to be raised up, and those nearer to the surface the more so.

As the first example of the formation of an image by refraction, we shall consider the following:—

**Image of Visible Point formed by Refraction at a Plane Surface.**—Suppose the point P, inside a transparent medium of index $\mu$, to be seen by an eye at E, so that E N P is normal to the surface of the medium. Draw the ray P R close to the normal; and let it be refracted along R S. Produce S R to meet P N in $q$. Then the image of P,



Fig. 40.

since it is seen by a very narrow pencil along P N E, will be at the ultimate position of $q$ when P R is indefinitely close to P N. Now R P N, R $q$ N are equal to the angles of incidence (from inside) and refraction (outwards) at R. Thus we have—
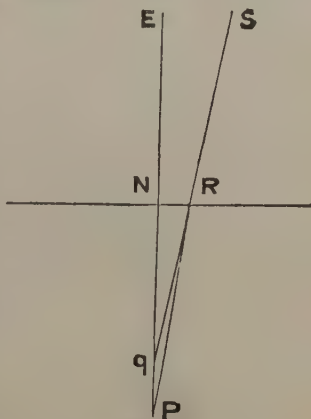
$$\sin \mathrm{RPN} = \frac{1}{\mu} \sin \mathrm{R}q\mathrm{N} ;$$
$$\frac{\mathrm{RN}}{\mathrm{RP}} = \frac{1}{\mu} \cdot \frac{\mathrm{RN}}{\mathrm{R}q} ;$$
$$\mathrm{RP} = \mu \mathrm{R}q.$$

In the limit this gives for the position of the image Q—

$$NQ = \frac{NP}{\mu}.$$

We have supposed the eye to look normally to the surface; but we shall see that, if this is not the case, no true image of P will be seen.

The result here found gives us a method of finding the refractive index of a thick plate of glass. With a reading microscope travelling vertically—that is, a microscope with a scale by which its vertical displacement can be measured—a reading is taken on a mark made on a horizontal piece of glass set so as to receive the plate. This mark corresponds to



P. The plate is now put on, and a reading is taken on the image of this mark, which corresponds to Q; and another on the top of the plate, which corresponds to N. From these we can find the thickness, and the apparent thickness of the plate; and the ratio of the former to the latter is the refractive index.

A similar method may be used to find the refractive index of a liquid.

**Displacement produced by Plate looked through normally.** — Let the object O (Fig. 41) be looked at through the plate of thickness $t$, normally to the plate. Let O be at distance $d$ from the near face of the plate. Let $\mu$ be the refractive index of the plate.

FIG. 41.

O forms, by the first refraction at B, an image at I′, so that BI′ $= \mu d$. I′ forms, by the second refraction at A, an image, I, so that—

$$AI = \frac{AI'}{\mu} = \frac{\mu d + t}{\mu} = d + \frac{t}{\mu}.$$

Thus $BI = d + \frac{t}{\mu} - t$; and $OI = t\left(1 - \frac{1}{\mu}\right).$
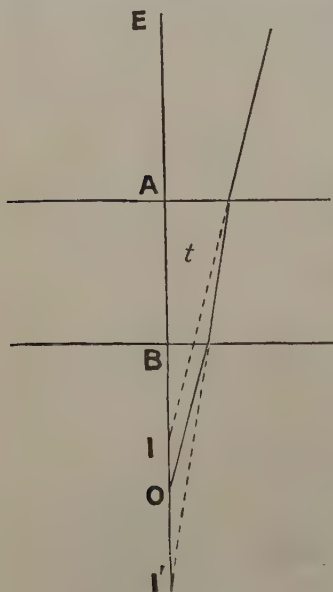
Thus at whatever distance the object is from the plate, it appears to be displaced towards it by a distance $t\left(1 - \dfrac{1}{\mu}\right)$.

**Shifting of Ray produced by Plate.**—Let A B C D (Fig. 42) be a ray passing through a plate of thickness $t$.

Draw C E perpendicular to A B produced. C E is the amount of shifting.

$$CE = CB \sin (i - r)$$
$$= \frac{t \sin (i - r)}{\cos r}.$$

**Refraction through a Prism.**—In geometry a prism is a solid figure contained by planes (forming its faces) which are all parallel to one straight line, or the intersections of which (forming the edges of the prism) are all parallel. The number of the faces may be any whatever, and the ends may be any whatever; or, indeed, the prism is not usually regarded as having ends at all. For optical purposes the prism will be taken of triangular section. The ends are generally perpendicular to all the faces; and, in general, only two of the faces are employed, so that only

FIG. 42.



FIG. 43.

E

these two need be polished surfaces. The edge where these two faces meet is called the **refracting edge** of the prism. We may call these two faces the refracting faces. The dih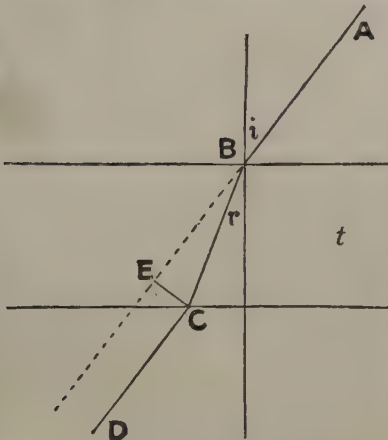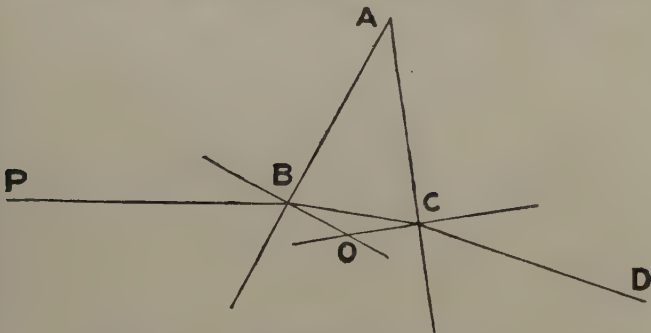edral angle contained by these faces is called the **refracting angle** of the prism. Any section of the prism made by a plane perpendicular to its faces is called a **principal section**. So that all principal sections are equal in all respects and parallel. And the angle made by the traces of the refracting faces in a principal section is equal to the refracting angle. Consider a ray of light in the plane of a principal section striking one face of a prism so that it emerges from the other face. It will continue in the same plane on entering the prism and on emerging. The figure represents the path of the ray. If $\phi_1$, $\phi_2$ are the deviations at the first and second faces, the entire deviation $\delta$ is $\phi_1 + \phi_2$.

Let $\mu$ be the refractive index of the prism. Let $i_1$, $i_2$ be the angles which the ray makes with the normals at B and C outside the prism; and $r_1$, $r_2$ the angles it makes with these normals inside the prism. We have—

$$\sin i_1 = \mu \sin r_1,$$
$$\sin i_2 = \mu \sin r_2,$$
$$\phi_1 = i_1 - r_1; \ \phi_2 = i_2 - r_2; \ \delta = i_1 - r_1 + i_2 - r_2.$$

The angles $i_1$, $r_1$, $i_2$, $r_2$ may be called the angles of external incidence, internal refraction, internal incidence, and external refraction. Now suppose the prism set so that the angle of external incidence is $i_2$. Then the angle of internal refraction is $r_2$; and since the sum of the internal angles is constant, as may easily be seen by the figure, the angle of internal incidence is $r_1$, and the angle of external refraction is $i_1$. Thus the deviation is the same as before, and the prism now occupies just the same position with regard to the incident and emergent light as it occupied before with regard to the emergent and incident light respectively. These two positions may be called conjugate. Notice that the deviations for two conjugate positions are equal. The deviation for any third position whatever would be different from that produced in these, as experiment would show, or as the above-written equations would show.

Now, there is one position of the prism which is its own conjugate—the position in which the prism is symmetrical with regard to the incident and emergent light, or in which $i_1 = i_2$; $r_1 = r_2$; and the triangle A B C is isosceles. For a given direction of the incident ray, P B, it is clear that a pair of

conjugate positions would be found by turning the prism to opposite sides of this position, one being formed by increasing $i_1$ and the other by diminishing $i_1$. Thus, as the prism is turned from this symmetrical position, the deviation either increases, whichever be the sense of turning, or else it decreases, whichever be the sense of turning. This symmetrical position, then, is the position giving either the minimum deviation of the ray or the maximum deviation. Experiment shows that *in the symmetrical position the deviation of the ray is a minimum.*

This is a very important result, and may also be proved by means of the equations—

$$\sin i_1 = \mu \sin r_1,$$
$$\sin i_2 = \mu \sin r_2,$$
$$\delta = i_1 + i_2 - r_1 - r_2.$$

From the figure $r_1 + r_2$ is supplementary to the angle at O, and therefore equal to the refracting angle, A, of the prism, Thus to see whether $\delta$ becomes a maximum or a minimum, we must see whether $i_1 + i_2$ becomes a maximum or a minimum.

Now consider the equation—

$$\sin i = \mu \sin r \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let $di$ and $dr$ be corresponding small increments in the angles $i$ and $r$, so that—

$$\sin (i + di) = \mu \sin (r + dr). \quad . \quad . \quad (2)$$

Subtracting (1) from (2), we get—

$$2 \cos \left( i + \frac{di}{2} \right) \sin di = 2\mu \cos \left( r + \frac{dr}{2} \right) \sin dr,$$

or, making $di$ and $dr$ indefinitely small—

$$\cos i \, di = \mu \cos r \, dr$$

—a result which comes from (1) by the differential calculus immediately.

$$di = \frac{\mu \cos r}{\cos i} dr$$
$$= \frac{\mu \cos r}{\sqrt{1 - \mu^2 \sin^2 r}} dr.$$

The square of the coefficient of $dr$ is—

$$\frac{\mu^2 \cos^2 r}{1 - \mu^2 + \mu^2 \cos^2 r} = \frac{1}{1 - \dfrac{\mu^2 - 1}{\mu^2 \cos^2 r}}.$$

Now, the greater $r$ is, the less is $\mu^2 \cos^2 r$, the greater is $\frac{\mu^2 - 1}{\mu^2 \cos^2 r}$; the less is the denominator of the above fraction, and the greater is the fraction.

Thus a given small variation in the direction of the ray in the denser medium causes a greater variation in that of the ray in the other medium, the greater the angle which the first ray makes with the normal. Or, again, the variation of $r$ from $r$ to $r + x$, where $x$ is a small positive angle, causes a greater increase in $i$ than the variation of $r$ from $r - x$ to $r$, or than the decrease in $i$ caused by the variation of $r$ to $r - x$. This result is interesting in itself.

To apply it to the case before us. Suppose $r_1$ and $r_2$ each to have the value $\frac{A}{2}$, as is the case for the symmetrical position. Then a small increment in $r_1$ must be accompanied by an equal decrement in $r_2$; and the resulting increment in $i_1$ is greater than the resulting decrement in $i_2$. So that a variation from the symmetrical position such as to increase $i_1$ causes $i_1 + i_2$, and therefore also the deviation, to increase; and the same would follow from a variation in the opposite sense. Thus the deviation in this position is a minimum. This position is generally called **the position of minimum deviation.**

**Expression for Refractive Index of Prism.**—Suppose the refracting angle A of the prism, and the angle of minimum
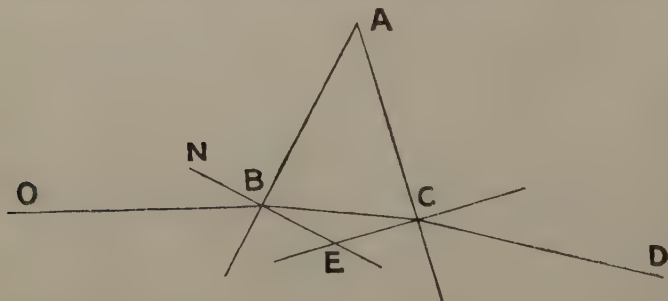


FIG. 44.

deviation, D, are known. Let the figure represent the position of minimum deviation, O B C D the ray. Then the angles $i_1$, $i_2$ are equal, and so are $r_1$, $r_2$. The triangles ABC, EBC are both isosceles. And $\mu = \dfrac{\sin \text{OBN}}{\sin \text{EBC}}$.

Now, the deviation at B is half the entire deviation.

$$\therefore \frac{D}{2} = OBN - EBC \, ;$$

$$OBN = \frac{D}{2} + EBC.$$

And EBC + ECB = supplement of E = A.

$$\therefore EBC = \frac{A}{2} \, ;$$

$$\mu = \frac{\sin \dfrac{D + A}{2}}{\sin \dfrac{A}{2}} \, .$$

This is a very important expression for the refractive index, and is employed in determining it in practice, by the most accurate method.

When the angle of the prism is very small, we can obtain a useful expression for D. For we then have—

$$\mu = \text{limiting value of } \frac{\sin \dfrac{D + A}{2}}{\sin \dfrac{A}{2}}$$

$$= \frac{D + A}{A}.$$

Thus $D = A(\mu - 1)$.

Consider a small divergent pencil refracted through a prism, very near to its edge, in the plane of a principal section, and at minimum deviation.

The divergence of the pencil at right angles to the plane of principal section is unaltered on passing out of the prism. For consider a ray in the prism at right angles to the edge, but not in the principal section. The position of this ray determines, and determines in precisely the same manner, the positions of the entering and emerging rays. Thus, since the ray we have considered inside the prism is situated in just the same way with regard to the two faces, the entering and emerging rays are situated in just the same way on the two sides of the prism, and the ray on emerging makes the same angle with the plane of principal section as on entering. Thus the divergence at right angles to this plane is unaltered.

The divergence in the plane of principal section is

unaltered.    For since the direction of the pencil is such as to
give minimum deviation, a small alteration in its direction will
produce no alteration, practically, in the deviation.    Thus two
rays in the plane of principal section will have the same
divergence on emergence as on entry, the limiting ratio of the
angle between the rays on emergence to the angle between them
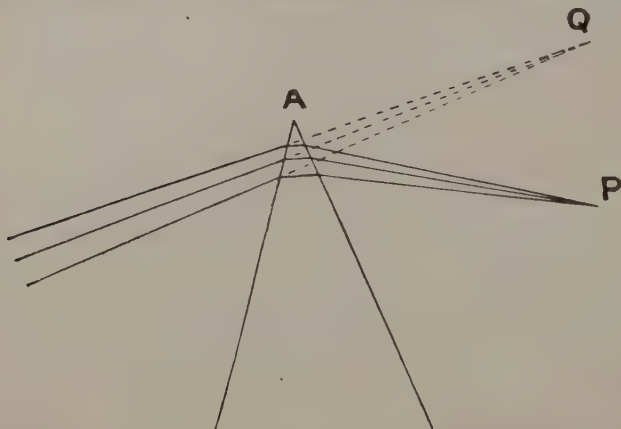


FIG. 45.

on entry, when the latter becomes indefinitely small, being unity.
Thus the divergence in the plane of principal section is unaltered.

It follows that the pencil emerging from the prism will be a
pencil diverging apparently from a single point in the plane of
principal section, and this point, Q, is at the same distance from
the prism where the rays enter it, or from A, as is the point
P, from which they originally come.

Thus a visible point, or a small object, at P, gives, by means
of a pencil or pencils in the position of minimum deviation, a
true virtual image at Q.    It should be noticed, as in the cases
of the mirror, that all the rays of a large pencil from P would
not pass through Q, this point being only the limiting position
of the intersection of these rays when they are taken indefinitely
near to each other and to the ray in the position of minimum
deviation.

It could be shown in just the same manner that a pencil
converging to Q, that is, so as to give a virtual object at Q,
would form, after refraction, a real image at P.

The points P and Q are **conjugate foci** with respect to
the prism.

### LENSES.

A **lens** is a portion of a refracting medium whose boundary surfaces are, in general, portions of surfaces of revolution having a common axis : this axis is called the axis of the lens.

If the surfaces do not meet, we may suppose the boundary of the medium to be completed by a portion of a cylinder having the same axis as the lens, but these portions will have nothing to do with the ordinary action of the lens.

We shall consider lenses having for boundary surfaces portions of spheres. The axis of such a lens is the straight line joining the centres of the spheres of which the boundary surfaces are parts.

One of the bounding surfaces may be a plane, as a particular case of a sphere having an infinitely large radius. Then the axis of the lens is the straight line drawn from the centre of the spherical surface perpendicular to the plane.

The six general forms of lenses are shown in the accom-
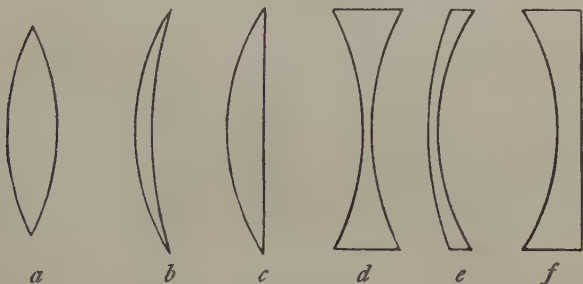


Fig. 46.

panying figure. They are sometimes called by the following names :—

  *a*, double convex ;
  *b*, convexo-concave, concavo-convex, or meniscus ;
  *c*, convexo-plane, or plano-convex ;
  *d*, double concave ;
  *e*, convexo-concave, or concavo-convex ;
  *f*, concavo-plane, or plano-concave.

The order of the words combined to name the lens refers to the order in which the light falls on the two surfaces. The surface on which the light first falls is called the *front*, and the other the *back* surface.

A more important classification is as follows :—

Lenses **thickest** in the middle are called **convex**, or **convergent.**

Lenses **thinnest** in the middle are called **concave**, or **divergent.**

To lead up to the refraction of light through lenses, let us first consider the following case : Suppose a small pencil of light to fall on the spherical boundary of a refracting medium, along a radius of the sphere : to find the focus of the pencil after refraction.

We shall make the same convention with regard to signs as was made in the case of reflexion. Distances will be measured along a radius from the surface of the medium, and those measured in the direction opposite to the incident light will be considered positive; those measured in the other direction, negative.

Let the boundary surface be represented by the circular arc A R in the figure, its radius A O being $r$, so that $r$ is, in this
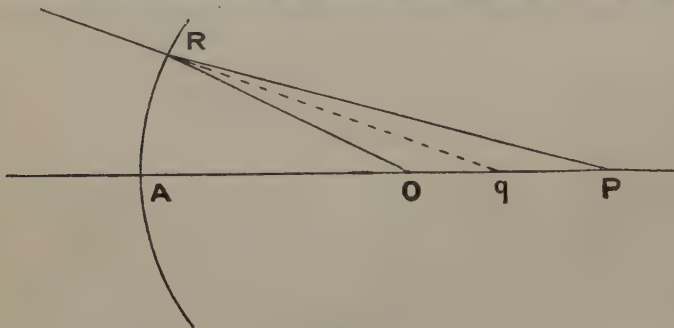


FIG. 47.

figure, positive. The refracting medium, of index $\mu$, is to the left of the surface.

P is a visible point on the radius through A. Let a ray from P meet the surface in R.

O R is normal to the surface at R. Thus the refracted ray at R lies in the plane O R P. And, supposing $\mu$ greater than I, its production backward lies between R O and R P. Let this ray meet A O P in $q$.

As the ray P R approaches to the line P A, and ultimately coincides with it, $q$ moves up to a definite limiting position : call this Q. Let $AP = u$, $AQ = v$. These distances are both positive in the figure, so that $u$ and $v$ are both positive quantities.

We have from the figure the relation—

$$\sin ORP = \mu \sin ORq.$$

To get a relation among the lengths of lines, we write this—

$$\sin ROP \cdot \frac{OP}{RP} = \mu \sin ROq \cdot \frac{Oq}{Rq}.$$

$$\text{Thus } \frac{OP}{RP} = \mu \frac{Oq}{Rq}.$$

In the limit this becomes—

$$\frac{OP}{AP} = \mu \frac{OQ}{AQ}.$$

Or—

$$\frac{u-r}{u} = \mu \frac{v-r}{v};$$

$$v(u-r) = \mu u(v-r).$$

Divide throughout by $uvr$, and we get—

$$\frac{1}{r} - \frac{1}{u} = \mu \left( \frac{1}{r} - \frac{1}{v} \right).$$

Or, we may write —

$$\frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{r}.$$

In the case we have taken $u$, $v$, $r$ are all positive; but the formula would be found to be universally true, whatever be the signs of these quantities. Let us, for example, suppose
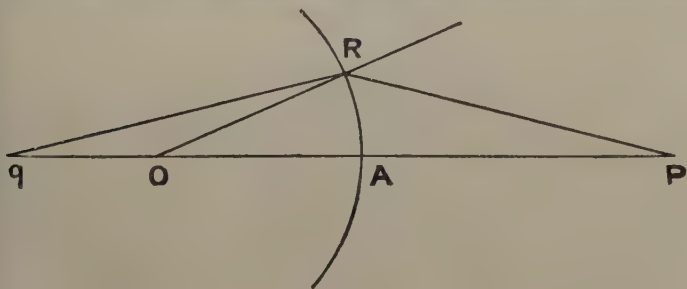


FIG. 48.

the bounding surface of the medium to be convex. Then it may be seen that the refracted rays may meet the radius through P on either side of A. Suppose they meet it on the left of A, which is the case for positions of P far from A.

Since PRO is supplementary to the angle of incidence, and $_q$RO is the angle of refraction, the figure gives—

$$\sin \text{PRO} = \mu \sin q\text{RO}.$$
$$\text{Thus } \sin \text{ROP} \cdot \frac{\text{OP}}{\text{RP}} = \mu \sin \text{RO}q \cdot \frac{\text{O}q}{q\text{R}};$$
$$\frac{\text{OP}}{\text{RP}} = \mu \frac{q\text{O}}{q\text{R}}.$$

In the limit this becomes—

$$\frac{\text{OP}}{\text{AP}} = \mu \frac{\text{QO}}{\text{QA}}.$$

In this, of course, the mere absolute magnitudes of the lengths involved are meant.

Now—

$$\text{OP} = \text{OA} + \text{AP} = -r + u,$$

since $-r$ is a positive quantity, and $r$ is in numerical magnitude equal to OA.

$$\text{AP} = u,$$
$$\text{QO} = \text{QA} - \text{OA} = -v + r,$$
$$\text{QA} = -v.$$

Therefore the above relation becomes—

$$\frac{u-r}{u} = \mu \frac{-v+r}{-v} = \mu \frac{v-r}{v}.$$

This is the same relation as before, and gives, as before—

$$\frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{r}.$$

If a small pencil of light along A O in the medium has Q for its focus, it will, on emergence, have P for its focus; since the paths of the rays are reversible. P and Q may be called **conjugate foci.**

The above formula is useful in itself; and would be used to find, for instance, the apparent position of a point, when we know the real position, or *vice versâ*, the point being embedded in a refracting medium with a spherical boundary, and looked at normally to the surface. Thus we can optically determine the thickness of a lens if we know its refractive index, and the curvature of one surface, and if we can find the position of the image of a mark on the other surface where the axis meets it when looked at through the first surface along the axis.

The formula has, however, a more important application, as we now proceed to show.

Suppose we have a lens whose thickness may be neglected. Let the radii of its front and back surfaces be $r$ and $s$, regard being had to the sign. In the figure drawn both $r$ and $s$ are
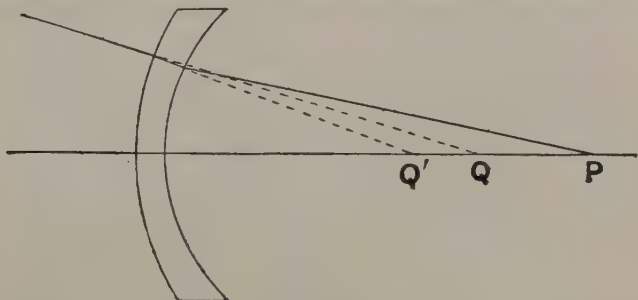


Fig. 49.

positive. Suppose there is a small axial pencil with focus at P, a point at a distance, $u$, from where the axis meets the lens. Since the thickness of the lens is neglected, we may suppose the axis to meet it in a point; and distances will be measured along the axis from this point. Let this pencil give rise, on entering the lens, to a pencil with focus Q', at distance $v'$ from the lens; and let this, on emerging, give rise to a pencil with focus at Q, at distance $v$ from the lens.

Then from refraction of the pencil into the lens we have—

$$\frac{\mu}{v'} - \frac{1}{u} = \frac{\mu - 1}{r}. \quad \ldots \quad (1)$$

And for the refraction out of the lens, since the corresponding refracting index is $\frac{1}{\mu}$, we have—

$$\frac{\frac{1}{\mu}}{v} - \frac{1}{v'} = \frac{\frac{1}{\mu} - 1}{s}.$$

Or—

$$\frac{1}{v} - \frac{\mu}{v'} = \frac{1 - \mu}{s}. \quad \ldots \quad (2)$$

Adding (1) and (2), we get—

$$\frac{1}{v} - \frac{1}{u} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right).$$

This very important formula gives the relation between the positions of a point on the axis of a lens and of its image as formed by the lens.

From the reversibility of the paths of the rays, it follows that a pencil on the left-hand side of the lens, having Q for focus, would give a pencil on the other side, with P for focus. The same thing would follow from the formula. The points P and Q are conjugate foci with respect to the lens.

We have met with four cases of conjugate foci. These were formed with respect to (1) spherical mirrors, (2) prisms, (3) refracting spherical surfaces, (4) lenses. In any case the existence of the conjugate foci is inferred at once from the reversibility of the rays. In any case a pair of conjugate foci has the following property: Suppose a pencil with P as focus, real or virtual; let this give a pencil with Q as focus, real or virtual; then a pencil with Q as focus of the same sort as that we have just supposed to be at Q will give a pencil with P as focus of the same sort as we supposed at P. Thus, for example, take the case of the prism, Fig. 45. A pencil with real focus at P gives a pencil with virtual focus at Q. A pencil coinciding with this latter, but convergent, whereas the latter was divergent, having a virtual focus at Q, will give rise to a pencil with real focus at P. A pencil with real focus at Q would give, by refraction through the prism, something quite different; in fact, a pencil with no point-focus at all.

Suppose that P (Fig. 49) goes to infinity, that is, that the pencil incident on the lens is a parallel one. The corresponding value of $v$ is called the focal length of the lens. It is denoted by $f$, and is given by—

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right).$$

The corresponding position of Q is called the principal focus of the lens, and we shall, in general, denote it by F.

The formula for the lens may now be written—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}.$$

For a *convex* lens, thickest in the middle, by examining all the cases that may occur, we see that $\frac{1}{r}$ is always algebraically less than $\frac{1}{s}$. For example, in the accompanying figure of a meniscus, with the light incident on the convex side, $r$ and $s$

are both negative, and $r$ is numerically smaller than $s$. There-
fore $\frac{1}{r}$ is numerically greater and algebraically less than $\frac{1}{s}$. And
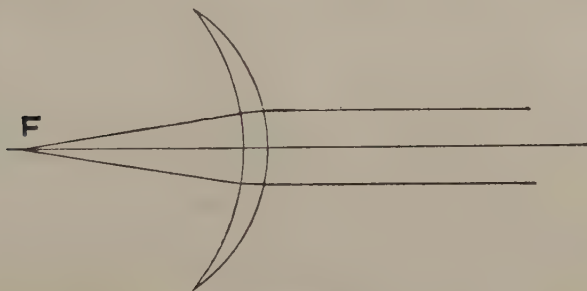


FIG. 50.

we should find $\frac{1}{r}$ algebraically less than $\frac{1}{s}$ in any case of a
convex lens.

Thus $\frac{1}{r} - \frac{1}{s}$ is a negative quantity;

$$\therefore f \text{ is } negative;$$

and F is on the *negative side* of the lens.

The rays which the parallel pencil gives on passing through
the lens *converge* to the point F, and really pass through it.

Thus F is a *real principal focus*.

From the formula—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f},$$

since $f$ is negative, it follows that $\frac{1}{v}$ is always algebraically less

than $\frac{1}{u}$. Let us consider the cases that can occur with this
condition. The figures show them.

(1) $u$ positive; $v$ may be positive; $v$ must then be greater
than $u$ (Fig. 51).

(2) $u$ positive; $v$ may be negative (Fig. 52).

(3) $u$ negative; $v$ must be negative, and numerically less
than $u$ (Fig. 53).

In any case it is seen that the pencil is more convergent
(or less divergent) after passing through the lens. Thus the
lens is called a *converging lens*.

For a *concave* lens, thinnest in the middle, by examining

all the cases that may occur, we see that $\frac{1}{r}$ is always alge-braically greater than $\frac{1}{s}$.
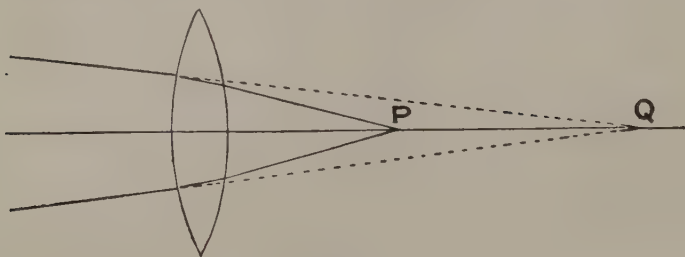


FIG. 51.
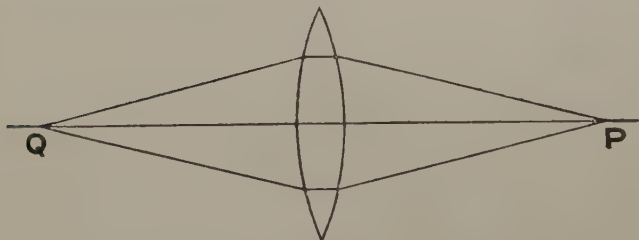


FIG. 52.
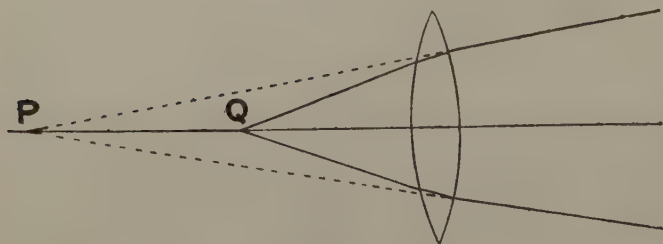


FIG. 53.

Thus $\frac{1}{r} - \frac{1}{s}$ is a positive quantity;

$$\therefore f \text{ is } positive;$$

and F is on the *positive side* of the lens.

The rays which the parallel pencil gives on passing through the lens appear to diverge from the point F, but do not really pass through it (Fig. 54).

Thus F is a *virtual principal focus.*
From the formula—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

since $f$ is positive, it follows that $\frac{1}{v}$ is always algebraically
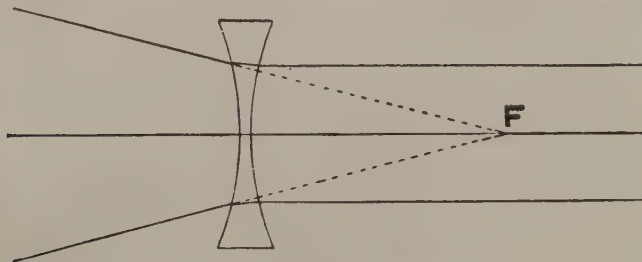


FIG. 54.

greater than $\frac{1}{u}$. Let us consider the cases that can occur
with this condition. The figures show them.

(1) $u$ positive; $v$ must be positive, and numerically less
than $u$ (Fig. 55).
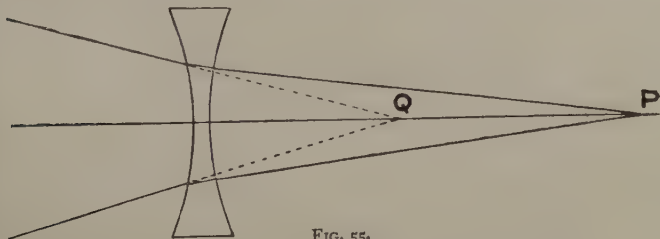


FIG. 55.

(2) $u$ negative; $v$ may be positive (Fig. 56).

(3) $u$ negative; $v$ may be negative; $v$ must then be
numerically greater than $u$ (Fig. 57).

In any case it is seen that the pencil is more divergent
(or less convergent) after passing through the lens. Thus the
lens is called a *diverging lens.*

It should be noticed that a convex lens behaves like a
concave mirror in having a real principal focus, and in con-
verging axial pencils; a concave lens behaves like a convex
mirror in having a virtual principal focus and in diverging
axial pencils.

Notice again, however, that both concave mirror and concave lens have positive focal lengths; and convex mirror and convex lens have negative focal lengths, according to our convention with regard to sign.
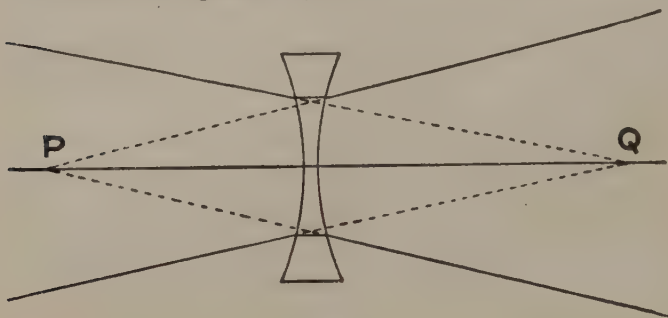


FIG. 56.

Any lens will have two principal foci, one for light falling on it from each side. They will be at equal distances from the lens. For if the radii of the surfaces for light falling on the lens from one side—right, say—are $r$ and $s$, the radii for
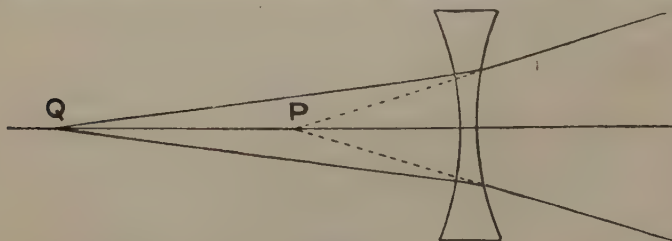


FIG. 57.

light falling on the lens from the left will be $-s$ and $-r$; so that the focal length, $f'$, for light from the left is given by—

$$\frac{1}{f'} = (\mu - 1) \left( \frac{1}{-s} - \frac{1}{-r} \right)$$

$$= (\mu - 1) \left( \frac{1}{r} - \frac{1}{s} \right) = \frac{1}{f}.$$

We shall denote the principal focus for light coming from the negative side by F'.

It should be noticed that a principal focus, on account of

the reversibility of rays, has also this property. A pencil of light having a principal focus of a lens for its focus will, after passing through the lens, become a parallel pencil, along the axis. Notice that the pencil must have the principal focus for a real or a virtual focus according as it is a real or a virtual focus of the lens.

We shall now consider the variations in position of the image as the position of the object is changed. We shall make use of the formula—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f},$$

or—

$$\frac{1}{v} = \frac{1}{f} + \frac{1}{u}.$$

1. *Convex lens*—

When $u = \infty$, $v = f$; *i.e.* object at infinity gives image at F.

As $u$ decreases in value from $\infty$ to the positive quantity $-f$, $v$ decreases algebraically from $f$ to $-\infty$. Thus, as object moves up from infinity to F′, image moves off from F to infinity on the left.

Object at F′ gives image at infinity.

As $u$ decreases from $-f$ to 0, $v$ decreases from $\infty$ to 0. Thus, as object moves from F′ up to the lens, the image, which is virtual, moves from infinity up to the lens.

It should be noticed that the object gives a real or a virtual image according as it is beyond or within the principal focus F′.

2. *Concave lens*—

When $u = \infty$, $v = f$; *i.e.* object at infinity gives virtual image at F.

As $u$ decreases from $\infty$ to 0, $v$ decreases from $f$ to 0. Thus, as the object moves up from infinity to the lens, the image moves up from F to the lens.

It should be noticed that the image is always virtual.

We have only considered the cases in which the object is real ; and the general conclusions drawn refer only to such cases. The cases in which the object is virtual may be considered in just the same way as the others by the help of the formula.

**Centre of Lens.**—Consider a lens of thickness $t$, so that AB $= t$ in the figure. Suppose a ray to pass through the lens so as to be undeviated. Then it must undergo equal and opposite deviations at entry and emergence, at the points

R and S. The radii to the surfaces at R and S must thus be parallel. Let these be $O_1R$, $O_2S$. S R lies in the plane with
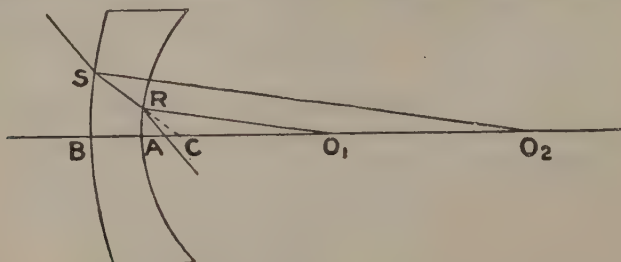


FIG. 58.

them, and so will meet the axis, say in the point C. C is a fixed point, as we shall show.

Let $AO_1 = r$; $BO_2 = s$. The two figures $RACO_1$, $SBCO_2$, are similar. Thus—

$$\frac{AC}{AO_1} = \frac{BC}{BO_2};$$
$$i.e. \frac{AC}{r} = \frac{AC + t}{s};$$
$$\therefore AC(s - r) = rt;$$
$$AC = \frac{rt}{s - r};$$
$$BC = \frac{st}{s - r}.$$

These values will always be found for the distances of C from A and from B measured to the right, whatever be the signs of $r$ and $s$.
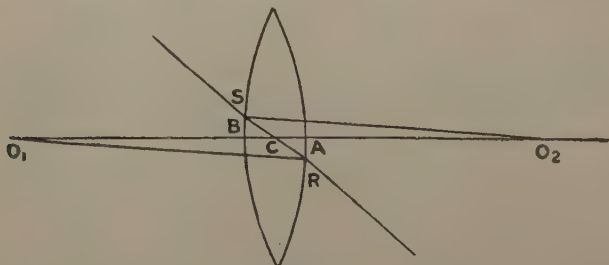


FIG. 59.

Take, for example, the double convex lens drawn in the above figure.

As before, we have—

$$\frac{AC}{AO_1} = \frac{BC}{BO_2};$$

$$BC = t - AC; \quad AO_1 = -r; \quad BO_2 = s.$$

$$\therefore \frac{AC}{-r} = \frac{t - AC}{s};$$

$$AC \cdot s = -rt + AC \cdot r;$$

$$AC = -\frac{rt}{s - r};$$

Thus the distance of C to the right of A is $\frac{rt}{s - r}$ (a negative

quantity, since $r$ is negative, $s$ and $t$ positive), and $BC = \frac{st}{s - r}$.

The point C is called *the centre of the lens.*

If we neglect the thickness, $t$, of the lens, $AC = BC = o$. Thus, for a thin lens the centre is *the point* at which the axis meets the lens; and a ray through this point will be undeviated.

An object situated on the axis of a lens, and all in a plane
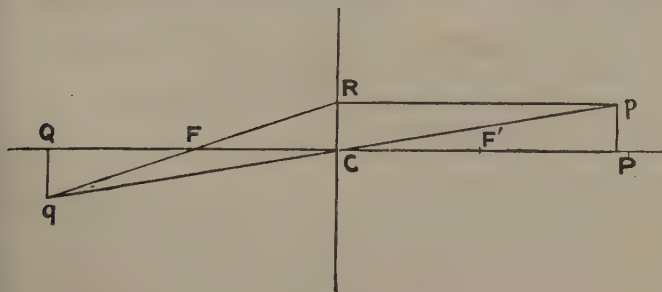


FIG. 60.

at right angles to the axis, and of very small dimensions, will have an image of itself formed on the axis, and similar to itself.

Consider a convex lens, and such an object $P\,p$, P being on the axis, and farther off than F'. The point $p$ sends out a ray, $p\,R$, parallel to the axis, which passes, after leaving the lens, through the focus F, very nearly, because $p$ is very near the axis. The ray $p\,C$, through the centre C, goes on undeviated. The point of intersection, $q$, of these two rays gives the focus of all the rays, which come from P, after passing

through the lens. Draw $q\,Q$ perpendicular to the axis. Then—

$$\frac{QF}{FC} = \frac{Qq}{CR} = \frac{Qq}{Pp} = \frac{QC}{CP}.$$

Thus Q is a fixed point, and the images of all points of the object are on the same plane through Q. And since—

$$\frac{Qq}{Pp} = \frac{QC}{CP},$$

it follows that the image is similar to the object.

It must be noticed that, for these results to hold, any point, as $p$, of the object must be so near the axis that the ray from it parallel to the axis may practically pass through F after deviation by the lens.

**Graphic Construction of Images.**—We shall now consider a method, similar to that used in the case of mirrors, of constructing the image, formed by a thin lens, of a small object on the axis. As in that case, we shall find the focus of a small pencil near the axis, after it passes through the lens by drawing two of its rays. And we can draw the three following rays :—

(1) The ray which passes through the centre of the lens : this continues in the same straight line.

(2) The ray which comes in parallel to the axis : this is continued through the focus F.

(3) The ray which comes in through the focus F′ : this is continued parallel to the axis.

We shall now consider the three different general cases
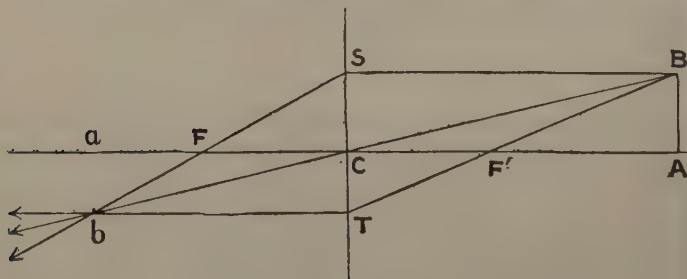


FIG. 61.

that may occur, giving the graphic construction for each case, the object being real.

I. *Convex Lens: Object beyond focus* F'.

Take A B, a small object in a plane perpendicular to the axis, with the point A on the axis (Fig. 61).

The rays B C, B S, B T, from the point B of the object, are respectively through the centre, parallel to the axis, and through the focus F'. They continue through the centre, through the focus F, and parallel to the axis. They give by their intersection the point *b*, the image of B. *b* is seen by means of a pencil of rays of which these are three. The image *a b* of the whole object is shown in magnitude and position by drawing *b a* perpendicular to the axis.

The image is *real* and *inverted*.

II. *Convex Lens: Object between lens and focus* F'.

The same rays as before are here drawn. The pencil they give, after passing through the lens, has now a virtual focus, *b*.
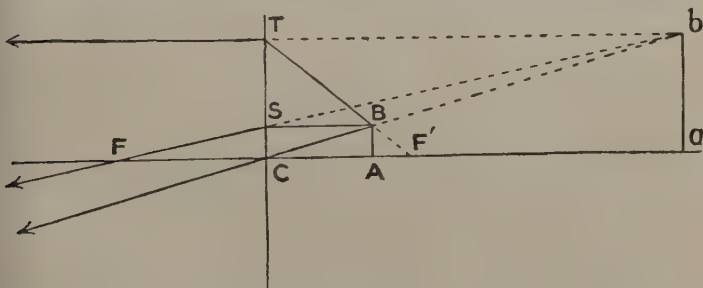


FIG. 62.

The rays *b* B, *b* S, *b* T have now no physical existence; but the action on the left-hand side of the lens, and the effect on the eye, are the same as if they had. The whole image *b a* is found as before.

The image is *virtual* and *erect*.

III. *Concave Lens.*

The construction follows the same lines as before. Notice, now, the interchange of positions of F and F'. Another figure should be drawn in which A is between C and F. It will be seen that there is no radical difference (Fig. 63).

The image is *virtual* and *erect*.

Remarks may be made on these cases similar to those made for mirrors.

The image is now real or virtual according as it is on the opposite side or on the same side of the lens as the object. In this respect the images formed by a mirror and lens differ.

For with a *lens* the rays from a given point pass really through
the image of the point when, and only when, the image is on
the *other* side of the lens; with a *mirror* they pass really
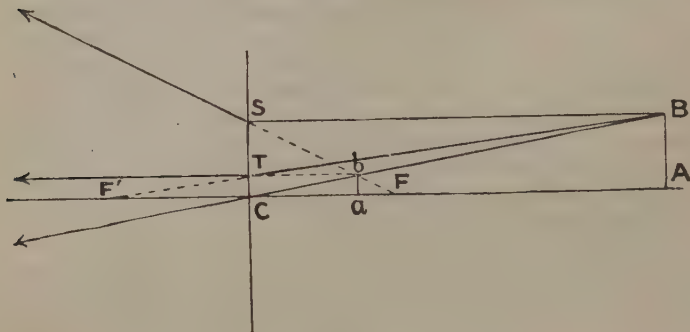


Fig. 63.

through the image when, and only when, the image is on the
*same* side.

The following statements have been made for mirrors, and
apply to mirrors and lenses :—

The image is erect or inverted according as it is on
the same side or on the opposite side of C from the
object.

The image is real when inverted, virtual when erect.

**Magnification.**—This term has just the same signification
here as for a mirror. Denote the magnification by *m*, as before.
Then from any of the figures just drawn—

$$m = \frac{ab}{AB}.$$

We shall now deduce some important expressions for *m*.
What follows is applicable to any one of the three figures. All
three should be referred to.

From similar triangles *abc*, ABC, we get—

$$m = \frac{ab}{AB} = \frac{Ca}{CA};$$

*i.e.* $m = \dfrac{\text{distance of image from lens}}{\text{distance of object from lens}}$ . . (1)

Taking the trace of the lens on the paper to be a straight
line at right angles to the axis, as is done in the case of the
mirrors, we get the following expressions :—

From similar triangles $ab\mathrm{F}$, CSF, we get—

$$m = \frac{ab}{\mathrm{AB}} = \frac{ab}{\mathrm{CS}} = \frac{a\mathrm{F}}{\mathrm{CF}}$$

*i.e.* $m = \dfrac{\text{distance of image from focus F}}{\text{focal length}}.$ . (2)

From similar triangles CTF′, ABF′, we get—

$$m = \frac{ab}{\mathrm{AB}} = \frac{\mathrm{CT}}{\mathrm{AB}} = \frac{\mathrm{CF}'}{\mathrm{AF}'}$$

*i.e.* $m = \dfrac{\text{focal length}}{\text{distance of object from focus F}'}$ . (3)

The numerical values merely of the quantities in these fractions are to be understood.

We can express $m$ in symbols in any of the following ways, from the above expressions, the numerical values of the fractions given being understood :—

$$m = \frac{u}{v} = \frac{v - f}{f} = \frac{f}{u + f}.$$

The expression number 3, involving the position of the object, and not of the image, is about the most important.

We shall draw some general inferences with regard to the magnification produced.

*Convex Lens.*—When the object is at a distance from the lens greater than $2f$, it is at a greater distance than $f$ from F′. Thus by (3) the image is *diminished*. When the object is at a distance from the lens less than $2f$, it is at a less distance than $f$ from F′. Thus the image is *magnified*.

*Concave Lens.*—The object is always at a greater distance than $f$ from F′ ; so that the image is always *diminished*.

We shall now enumerate, for reference, the characteristics of the image formed by either lens, and for all positions of the object, with regard to the following four particulars :—

(1) Position ;
(2) Whether real or virtual ;
(3) Whether erect or inverted ;
(4) Size.

*Convex Lens—*

Object at greater distance from lens than $2f$; image to left of F, real, inverted, diminished.

Object at distance $2f$; image at distance $2f$ on left, real, inverted, equal to object,

Object at distance between $f$ and $2f$; image to left of F, real, inverted, magnified.

Object between lens and F'; image to right of lens, virtual, erect, magnified.

*Concave Lens*—

Object in front of lens; image in front of lens, between lens and F, virtual, erect, diminished.

Throughout, the close resemblance of behaviour between a concave mirror and a convex lens, and between a convex mirror and a concave lens, should be noticed.

**Combination of Thin Lenses in Contact.**—Suppose any number of lenses of unappreciable thickness be placed so as to have the same axis and to be all in contact. Let the lenses be $n$ in number, and have focal lengths $f_1, f_2, \ldots f_n$. Suppose a small object, at distance $u$ from the combination, forms an image in the first at distance $v_1$; this forms an image in the second at distance $v_2$; and so on; so that an image is formed in the last at distance $v_n$.

Then—

$$\frac{1}{v_1} - \frac{1}{u} = \frac{1}{f_1};$$

$$\frac{1}{v_2} - \frac{1}{v_1} = \frac{1}{f_2};$$

$$\frac{1}{v} - \frac{1}{v_{n-1}} = \frac{1}{f_n}.$$

Therefore, by addition—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f_1} + \frac{1}{f_2} + \ldots + \frac{1}{f_n} = \Sigma\left(\frac{1}{f}\right).$$

Now, a single lens, if placed in the same position as the combination, would form an image of the object in the same position as this image, if its focal length, F, is given by—

$$\frac{1}{F} = \Sigma\left(\frac{1}{f}\right)$$

This lens is said to be **equivalent** to the given system of thin lenses.

If the lenses are separated by finite distances, $a_1$, $a_2$, etc., no single thin lens can be found so as in all cases to produce an image of a given small object in the same position as the lenses would. In this case the final position of the image is to be calculated by such equations as—

$$\frac{1}{v_1} - \frac{1}{u} = \frac{1}{f_1},$$

$$\frac{1}{v_2} - \frac{1}{v_1 + a_1} = \frac{1}{f_2},$$

and so on.

If we have two lenses, of focal lengths $f_1, f_2$, separated by an interval, $a$; and if $v$ is the distance from the second at which is formed a focus for *a parallel pencil* striking the first, then—

$$\frac{1}{v_1} = \frac{1}{f_1}$$

$$\text{and } \frac{1}{v} - \frac{1}{v_1 + a} = \frac{1}{f_2}$$

$$\therefore \frac{1}{v} = \frac{1}{f_1 + a} + \frac{1}{f_2}$$

$$v = \frac{f_2(f_1 + a)}{f_1 + f_2 + a}$$

This is the focal length of a lens which must be used instead of the combination, and placed in the position of the second, to bring the parallel pencil to the same focus as the combination.

**Determination of Focal Length.**—The focal length of a lens can be found by many methods. We shall suppose, here, that the thickness of the lens is inconsiderable.

Suppose a small pencil of light to fall on the lens axially from a focus, P, and to produce a conjugate focus, Q. Then, if the distances of P and Q from the lens can be measured, the focal length can be calculated by the formula—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}.$$

For a convex lens this method may be practised by fixing the lens in a holder on the optical bank, with its axis parallel to the bank, and setting up a small object, such as a pair of cross-wires, on another holder, so that this object can be moved along the axis of the lens. Then set a screen on the other side of the lens and fix it in the best position for receiving an image, formed by the lens, of the object. $u$ and $v$ can now be measured, and in this case it must be noticed that $v$ is negative.

For a concave lens with a real object we should always

have a virtual image. The method may be carried out by using a virtual object. Place a convex lens so as to converge a pencil of light to a point on its axis. Now place the concave lens in the way of this pencil, so that the rays do not reach this focus, but converge to one further off, being diverged by the concave lens. The distances from the lens, of these two foci, must be measured; they are respectively $-u$ and $-v$.

A concave lens may be combined with a convex of known focal length, $f_1$, and such that it forms with the concave a convex combination, whose focal length, F, can be found. Then the focal length, $f_2$, of the concave is found by—

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}.$$

For a convex lens a pencil of parallel rays, such as the rays from the sun, may be allowed to fall on the lens along its axis. The point of convergence of these rays can then be found by receiving them on a screen. This point is a principal focus of the lens, and its distance from the lens is, numerically, the focal length.

In any case we may use a virtual image if we can find its position. We may, for instance, find the focal length of a concave lens by letting parallel rays fall on the lens, giving a virtual focus; and then measure the distance of this focus from the lens. The position of a virtual image may be found as follows: A convex lens and screen are arranged at a fixed distance apart, and the distance of an object from the lens, so placed as to produce an image on the screen, is found. Then, if the lens and screen are placed so as to produce an
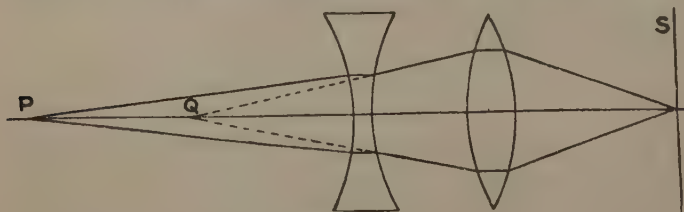


FIG. 64.

image on the screen of a given virtual image, the position of this image is known.

The focal length of the lens and its distance from the screen must be such as to allow it to get near enough to the virtual image in any case.

The accompanying diagram shows the way in which the arrangement is used to determine the position of the virtual image Q of an object, P, formed by a concave lens.

Another method, similar in principle to this, for doing the same thing, is to have, instead of convex lens and screen, a reading telescope : this consists simply of a convex lens, a pair of cross-wires, taking the place of S, at which the image is to be formed, and a lens of short focal length behind the cross-wires by which to see the image.

### EXAMPLES.

1. A horizontal plate of glass 0·3 in. thick is under a depth of 2·5 ins. of water : to an eye looking vertically downwards, find how far below the surface of the water a small spot on the under surface of the glass appears to be, the refractive indices of glass and water being 1·6 and 1·3.

2. The minimum deviation of a ray of light produced by passing through a prism of angle 60° 6′ 20″ is 42° 40′ 20″ : show how to use these results to determine the refractive index of the glass prism, and find it, having given—

$$\text{L sin } 51° 24' = 9·89294, \quad \text{L sin } 30° 4' = 9·69984,$$
$$\text{L sin } 51° 23' = 9·89284, \quad \text{L sin } 30° 3' = 9·69963,$$
$$\log 1·5610 = 0·19340, \quad \log 1·5600 = 0·19312.$$

(Lond. Int. Sci. Hons., 1886.)

3. Taking the refractive index from air to glass as $\frac{3}{2}$, draw an accurate picture of the path of a ray of monochromatic light, which falls at an incidence of 60° on the face of a prism, whose vertical angle is 30°. (Science and Art Advanced, 1894.)

4. Find the apparent position of a small object at the centre of a globe containing water, when seen by an eye outside ; the radius of the globe being 5 ins., and the refractive index of water $1\frac{1}{3}$.

5. Given a double concave lens of 5 cms. thickness, the radii of curvature of its faces being 15 and 20 cms. respectively : find the position of the image of a point on the axis 24 cms. from the nearer face. (Lond. Int. Sci. Hons., 1884.)

6. A small air-bubble in a sphere of glass 4 ins. in diameter, appears, when looked at so that the bubble and the centre of the sphere are in a line with the eye, to be 1 in. from the surface : what is its true distance ? ($\mu = 1\cdot5$.) (Lond. Int. Sci. Pass, 1887.)

7. An object is 20 feet from a screen : given two convex lenses respectively of 9 ins. and 18 ins. focal length, explain how you will obtain (1) an erect and magnified, (2) an inverted and magnified image of the object on the screen. (Lond. Int. Sci. Pass, 1886.)

# CHAPTER IV.

### *FOCAL LINES—ABERRATION—CAUSTICS.*

WE have considered, hitherto, cases of pencils reflected or refracted in such manners as to give, after reflexion or refraction, pencils with definite point-foci. We shall now see that, in general, the reflected or refracted pencil will not have a focus, that is, there is no one point through which all its rays pass, even in the limit when the pencil is taken indefinitely narrow, as there is in the special cases we have considered. These cases, however, which are the easiest to consider, are at the same time the most important in their bearing on practical applications of optics. The property of the pencil, that it has no point-focus, is called **astigmatism**. For an astigmatic pencil there are two regions, along its length, of maximum concentration of the rays, or of maximum intensity. These approximate, in the cases we shall have to consider, as the pencil is taken indefinitely narrow, to two straight lines in directions at right angles to each other and to the pencil. These lines are called **focal lines**. It will be convenient to consider one ray of the pencil as the **principal ray** round which the others are clustered. The points in which the principal ray is met by the focal lines are called the focal points. An astigmatic pencil may be illustrated by a number of stretched threads all lying close together, but not passing through a point, and made to pass through two straight slits at two different places along them, set at right angles to each other and to the general direction of the threads, care being taken that no thread is bent out of the straight line by the edge of a slit. It may happen, of course, that the focal lines of a pencil are virtual, so that they are lines through which the rays would pass if produced.

Suppose an eye so placed as to receive pencils coming from a visible point after they have been rendered astigmatic by reflexion or refraction. These pencils will give no true geometrical image of the point. And the assemblage of such pencils coming originally from the various points of an object give no true image of the object. Such a case is when an object is looked at under water obliquely to the surface, or when an object is viewed by oblique reflexion in a spherical

mirror. In such cases, as we know, a sort of image is seen; but this will be more or less blurred—the images of the various points will be situated between the focal lines of the corresponding pencils. This image will be more blurred the further apart the focal lines of each pencil are. The quantities of light coming from two neighbouring points of the object do not reach the eye as if they came from two distinct points, but will overlap and encroach upon each other, more or less, thus causing indistinctness.

Suppose a straight line, as object, giving a series of astigmatic pencils (by reflexion or refraction); and suppose the image, of the sort above mentioned, formed by them, to come out a straight line parallel to the common direction of all the focal lines, of one sort, of the pencils. Then the result is that we have a straight line (the image) from all points of which rays proceed to the eye; and there are no rays, coming originally from the object, reaching the eye that do not pass through points on this straight line. There is, therefore, formed, in this case, a distinct image of the object, with no confusion, the image coinciding with the assemblage of focal lines, of one sort, of the pencils which reach the eye. The various points of the image do not correspond, separately, with the various points of the object, so that if the various points of the object were to be distinguished from each other, since these have corresponding to them in the image short overlapping pieces, the image would be blurred; but regarding the object merely as a straight line, the image also comes out a definite straight line, and the light proceeding from it to the eye may be regarded as an assemblage of pencils proceeding from its various points as from foci, although the light in any one of these pencils did not come from a single point of the object.

The appearance presented to an eye will depend upon where the eye is placed. For instance, we have seen what the appearance is in the case of an eye looking at a small object under water normally to the surface of the water. The image thus seen is the only true geometrical image of this object formed by rays refracted through the surface, or the only true image that can be seen by an eye looking at the object through the surface. But if the object is viewed obliquely to the surface, a confused image will be seen by means of astigmatic pencils; and this image will be in a different position from the true image.

Suppose a narrow pencil, with P as focus, to fall on a plane

or spherical reflecting or refracting surface. Let P R be the principal ray of the pencil. Let P O be normal to the surface. All the rays of the reflected or refracted pencil (or deviated pencil) will meet P O. The section of this pencil by a plane
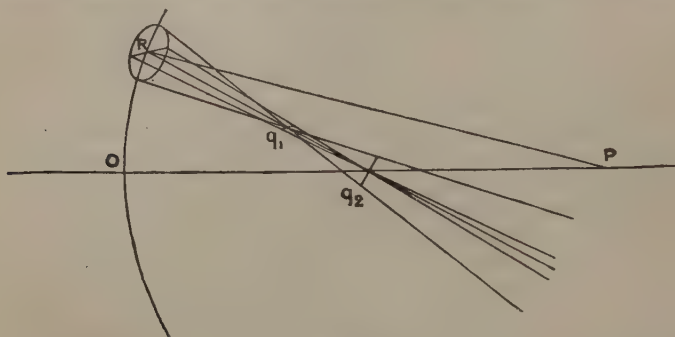


FIG. 65.

through the point where P O is met by the deviated ray from R and perpendicular to this ray is a very elongated figure of 8, approximating to a straight line as the pencil becomes very narrow. Denote it by $q_2$. This is called the **secondary focal line.**

All the rays of the pencil which are in any plane through P O will converge to or diverge from a point in this plane. So that all the rays of the pencil, being situated in a series of planes through P O, will converge to or diverge from a small straight line, $q_1$, at right angles to the plane P O R. This line is called the **primary focal line.**

The plane P O R is called the **primary plane.**

The plane through the deviated ray from R at right angles to the primary plane is called the **secondary plane.**

The primary and secondary focal lines are, then, at right angles to the primary and secondary planes.

We shall now consider, in detail, some of the pencils formed by reflexion and refraction of pencils with point-foci, and find the positions of their focal lines.

**Reflexion at Plane Surface.**—All the rays of a given pencil pass, after reflexion at a plane surface, accurately through the geometrical image of the focus of the given pencil. Thus no pencil will give a pencil having focal lines in this case. The reflected pencil will always have a point-focus.

**Refraction at Plane Surface.**—Let the pencil from P, with rays such as P R, P S, be refracted at a plane surface. The figure is drawn for refraction from a denser to a lighter medium.

Draw P O normal to the surface. The secondary focal line, $q_2$, is on P O.

To find the position of the primary focal line, $q_1$, let us consider two neighbouring rays P R, P S, in a plane with P O. Let the angles of incidence and refraction of these rays be $i$, $r$; $i + di$, $r + dr$. Let $\mu$ be the index of refraction.
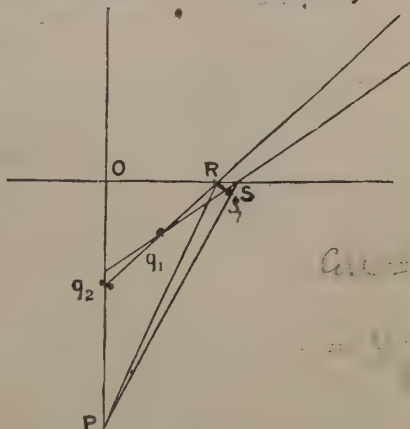
Let PR = $u$; $q_1$R = $v_1$; $q_2$R = $v_2$.

Now, $\angle$ RPS = $di$; $\angle$ R$q_1$S = $dr$.



Fig. 66.

Thus $di = \dfrac{\text{RS}\cos i}{u}$; $dr = \dfrac{\text{RS}\cos r}{v_1}$.

But since $\sin i = \mu \sin r$,

$$\therefore \cos i\, di = \mu \cos r\, dr,$$
$$\therefore \frac{\cos^2 i}{u} = \frac{\mu \cos^2 r}{v_1},$$
$$v_1 = \frac{\mu u \cos^2 r}{\cos^2 i}.$$

Again, OR = $u \sin i = v_2 \sin r$.

$$\therefore v_2 = \mu u.$$

From these results we have—

$$v_1 = v_2 \frac{\cos^2 r}{\cos^2 i}.$$

Thus $q_1$ is nearer to or further from the surface than $q_2$ according as $r$ is greater or less than $i$, that is, according as the refraction is into the lighter or denser medium.

**Reflexion at Spherical Surface.**—Let C be the centre of the sphere.

Let $CO = R$; $PR = u$; $Rq_1 = v_1$; $Rq_2 = v_2$.
Let the angles of incidence and reflexion at R be $i$.

$$\angle RCS = \angle RPS + \angle PRC - \angle CSP$$
$$= \angle RPS + \tfrac{1}{2}PRq_1 - \tfrac{1}{2}\angle PSq_1.$$
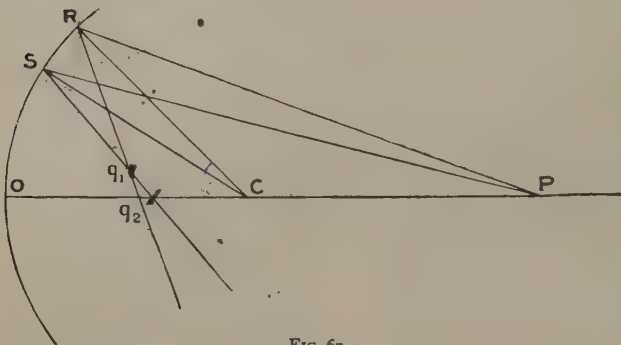


Fig. 67.

So $\angle RCS = \angle Rq_1S + \tfrac{1}{2}\angle PSq_1 - \tfrac{1}{2}\angle PRq_1$.

$$\therefore \angle RPS + \angle Rq_1S = 2\angle RCS;$$

$$i.e. \frac{RS \cos i}{u} + \frac{RS \cos i}{v_1} = \frac{2RS}{R};$$

$$\frac{1}{u} + \frac{1}{v_1} = \frac{2}{R \cos i}.$$

Let $\angle RCq_2 = \theta$.

$$\text{Then } \frac{R}{u} = \frac{\sin(\theta - i)}{\sin \theta};$$

$$\frac{R}{v_2} = \frac{\sin(\theta + i)}{\sin \theta}.$$

$$\therefore \frac{R}{v_2} + \frac{R}{u} = 2 \cos i,$$

$$\frac{1}{v_2} + \frac{1}{u} = \frac{2 \cos i}{R}.$$

Or this may be proved as follows :—

$$\triangle Rq_2P = \triangle Rq_2C + \triangle RCP;$$
$$\therefore v_2u \sin 2i = v_2R \sin i + Ru \sin i;$$
$$\therefore \frac{1}{v_2} + \frac{1}{u} = \frac{2 \cos i}{R}.$$

**Refraction at Spherical Surface.**—Let the angles of incidence and refraction at R and S be $i$, $r$, $i + di$, $r + dr$.
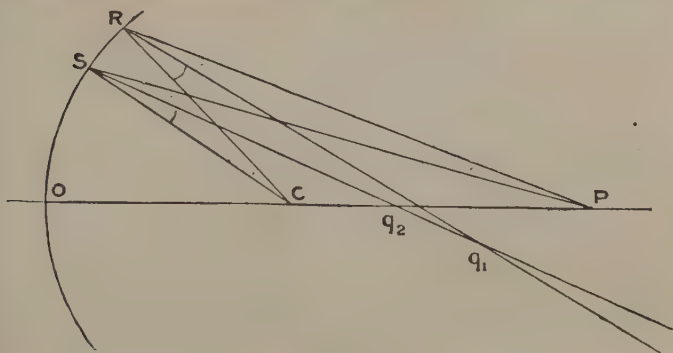


FIG. 68.

Let $CO = R$; $PR = u$; $Rq_1 = v_1$; $Rq_2 = v_2$.

$$\text{Then } di = PSC - PRC$$
$$= RPS - RCS;$$
$$dr = q_1SC - q_1RC$$
$$= Rq_1S - RCS.$$

$$\text{Thus } di = \frac{RS \cos i}{u} - \frac{RS}{R};$$

$$dr = \frac{RS \cos r}{v_1} - \frac{RS}{R}.$$

$$\text{But } \sin i = \mu \sin r;$$
$$\therefore \cos i \, di = \mu \cos r \, dr.$$

$$\text{Thus } \frac{\cos^2 i}{u} - \frac{\cos i}{R} = \frac{\mu \cos^2 r}{v_1} - \frac{\mu \cos r}{R};$$

$$\therefore \frac{\mu \cos^2 r}{v_1} - \frac{\cos^2 i}{u} = \frac{\mu \cos r - \cos i}{R}.$$

Let $\angle RCO = \theta$.

$$\text{Then } \frac{R}{u} = \frac{\sin (\theta - i)}{\sin \theta} = \cos i - \cot \theta \sin i;$$

$$\frac{R}{v_2} = \frac{\sin (\theta - r)}{\sin \theta} = \cos r - \cot \theta \sin r.$$

Multiply the second of these equations by $\mu$; subtract the first from it, and divide by R, and we get—

G

$$\frac{\mu}{v_2} - \frac{1}{u} = \frac{\mu \cos r - \cos i}{R}.$$

Or this may be proved as follows :—

$$\triangle \text{RCP} = \triangle \text{RC}q_2 + \triangle \text{R}q_2\text{P} ;$$

$$\therefore \text{R}u \sin i = \text{R}v_2 \sin r + uv_2 \sin (i-r).$$

Divide by $uv\text{R} \sin r$, and remember $\dfrac{\sin i}{\sin r} = \mu$. Thus—

$$\frac{\mu}{v_2} - \frac{1}{u} = \frac{\mu \cos r - \cos i}{R}.$$

An astigmatic pencil may be produced by reflexion or refraction of a pencil already astigmatic ; and we may require to know the focal lines of the pencil so produced.



Fig. 69.

Suppose a pencil with focus P to pass through a plate of glass of index $\mu$. Let the pencil after incidence at R S have the focal lines $q'_1$ and $q'_2$. Let this pencil emerge at T U.

Let P M N be the normal to the plate, or the axis, through P. All the rays incident at T U come from points on the axis at $q'_2$, so that they will go on after emergence from points on the axis. Thus a focal line, the secondary, will be formed on the axis at $q_2$.

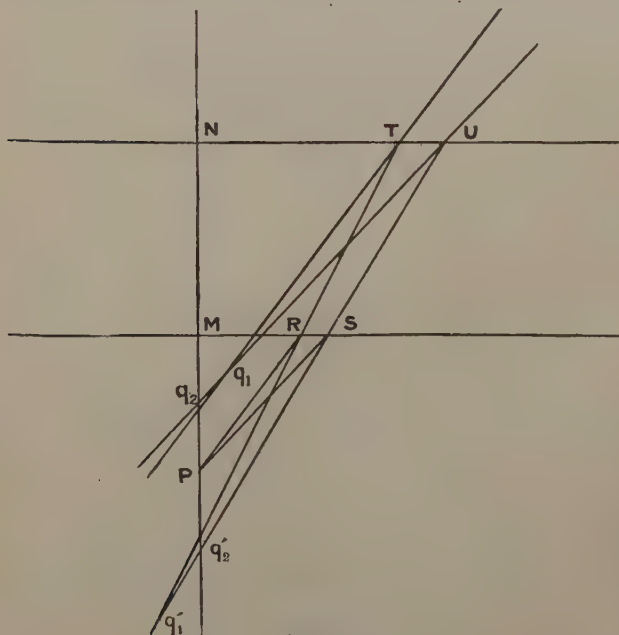All the rays of the pencil in any plane containing the axis, that is, coming from any given point of $q'_1$, will proceed, after emergence, from some point in this plane. The position of this point we may calculate from that of $q'_1$ by a formula already given at p. 79 for refraction at a single surface. The assemblage of such points in all the planes through the axis, and containing the various points of $q'_1$, will be the primary focal line $q_1$.

Let $i$ and $r$ be the angles of incidence and refraction at R, and of refraction and incidence at T. Then we have—

$$\mathrm{T}q_1 = \frac{1}{\mu} \cdot \frac{\cos^2 i}{\cos^2 r} \cdot \mathrm{T}q'_1$$

And—

$$\mathrm{T}q'_1 = \mathrm{R}q'_1 + \mathrm{T R} = \mu \frac{\cos^2 r}{\cos^2 i} \cdot \mathrm{R P} + \frac{t}{\cos r};$$

$$\therefore \mathrm{T}q_1 = \mathrm{R P} + \frac{t \cos^2 i}{\mu \cos^3 r};$$

And—

$$\mathrm{T}q_2 = \frac{1}{\mu} \mathrm{T}q'_2$$

$$= \frac{1}{\mu}(\mathrm{R}q'_2 + \mathrm{T R})$$

$$= \mathrm{R P} + \frac{t}{\mu \cos r}.$$

If the thickness of the plate may be neglected, $\mathrm{T}q_1 = \mathrm{T}q_2$. Thus in this case $q_1$ and $q_2$ coincide, and the emergent pencil has a point-focus. Thus by refraction through a thin plate a true geometrical image is practically formed.

Suppose a pencil to be refracted through a prism very near to its edge, and in a plane of principal section. Let the angles of incidence and refraction at the first and second faces be $i$, $r$; $r'$, $i'$. Let P, the focus of the pencil before entering the prism, be at distance $u$ from the edge.

At the first refraction let the focal lines $q'_1$, $q'_2$ be formed at distances $v'_1$, $v'_2$ from the edge. The pencil having these

focal lines undergoes refraction at the second surface, and
we wish to investigate the form of the pencil so produced.
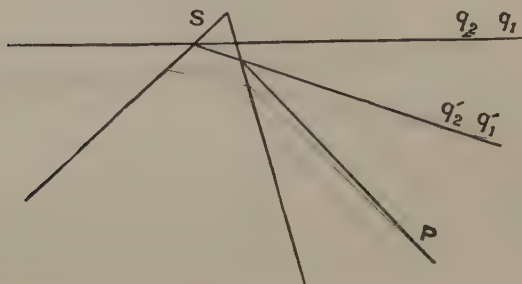


FIG. 70.

$q'_2$ is an assemblage of points in the plane of the paper.
A pencil going from one of these points in the direction $q'_2$ S
has, after refraction at S, a secondary focal line $q_2$. All the
rays of the given pencil from any one point of $q'_2$ must,
since they are in a plane at right angles to that of the paper,
after refraction, meet in a point of $q_2$. Thus all the rays from
all the points of $q'_2$ meet in points in the plane of the paper
at $q_2$. Thus $q_2$ is the secondary focal line of our astigmatic
pencil after refraction at S. Thus this focal line depends
only on the position of $q'_2$, and not on that of $q'_1$; and has
the same position as that of a pencil with focus at $q'_2$.

In the same way the rays from the various points of $q'_1$,
which is at right angles to the plane of the paper, give, by
their intersection after refraction, the various points of $q_1$, the
primary focal line, and this is the same as that of a pencil
with focus at $q'_1$.

We can thus find the positions of $q_1$ and $q_2$. By the
formulæ at p. 79—

$$v_1 = \frac{1}{\mu} \cdot \frac{\cos^2 i'}{\cos^2 r'} \cdot v_1'$$

$$= \frac{1}{\mu} \cdot \frac{\cos^2 i'}{\cos^2 r'} \cdot \mu \frac{\cos^2 r}{\cos^2 i} u$$

$$= \frac{\cos^2 i' \cos^2 r}{\cos^2 r' \cos^2 i} . u.$$

$$v_2 = \frac{1}{\mu} v_2'$$

$$= \frac{1}{\mu} \mu u$$

$$= u.$$

For the pencil at emergence to diverge from a point, we must have $v_1 = v_2$. Thus we must have $i = i'$, $r = r'$, or the pencil must be in the position of minimum deviation. In this case the distance of its focus at emergence from the edge is $u$. These results coincide with what we have already seen at p. 62.

**Pencil passing obliquely and centrically through Thin Lens.**—When a pencil passes centrically, as P R S T, that
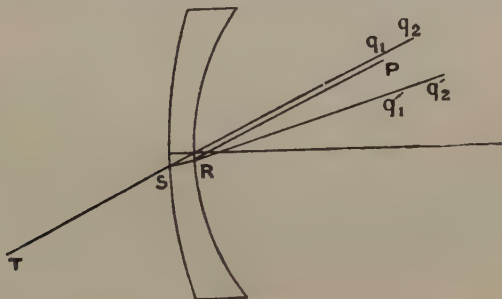


FIG. 71.

is, so that the part R S in the lens passes through the centre, it is undeviated. For a thin lens, the points R, S correspond with each other and with the centre; and we may suppose distances to be measured from any of these three points indifferently.

Let the surfaces of the lens have radii R, S; and let $\mu$ be the index of refraction. Let the pencil P R S T have P for focus, and be inclined at an angle, $i$, to the axis of the lens; and let $r$ be angle of refraction into the lens. Suppose this pencil has, on entering the lens, the focal lines $q_1'$, $q_2'$; and on emergence the focal lines $q_1$, $q_2$.

Let $RP = u$; $Rq_1' = v_1'$; $Rq_2' = v_2'$; $Sq_1 = v_1$; $Sq_2 = v_2$.
Then we have—

$$\frac{\mu \cos^2 r}{v_1'} - \frac{\cos^2 i}{u} = \frac{\mu \cos r - \cos i}{R}.$$

And since the index of refraction out of the lens is $\frac{1}{\mu}$ —

$$\frac{\cos^2 i}{\mu v_1} - \frac{\cos^2 r}{v_1'} = \frac{1}{\mu} \frac{\cos i - \cos r}{S}.$$

Or—

$$\frac{\cos^2 i}{v_1} - \frac{\mu \cos^2 r}{v_1'} = \frac{\cos i - \mu \cos r}{S}.$$

Adding and dividing by $\cos^2 i$, we get—

$$\frac{1}{v_1} - \frac{1}{u} = \frac{\mu \cos r - \cos i}{\cos^2 i} \left( \frac{1}{R} - \frac{1}{S} \right).$$

Again—

$$\frac{\mu}{v_2'} - \frac{1}{u} = \frac{\mu \cos r - \cos i}{R};$$

$$\text{and } \frac{1}{v_2} - \frac{\mu}{v_2'} = \frac{\cos i - \mu \cos r}{S};$$

$$\therefore \frac{1}{v_2} - \frac{1}{u} = (\mu \cos r - \cos i) \left( \frac{1}{R} - \frac{1}{S} \right).$$

When $i$ is small, $\cos^2 i$ is very nearly equal to unity, and the difference between $v_1$ and $v_2$ is very small. Thus the focal lines are very near together, and a good approximation to a geometrical image is obtained for points near the axis.

In this case, $i$ and $r$ being small quantities of the first order, $\cos i$ and $\cos r$ differ from unity by small quantities of the second order. So that we have the equations, correct to small quantities of the first order,—

$$\frac{1}{v_1} - \frac{1}{u} = \frac{1}{v_2} - \frac{1}{u} = (\mu - 1) \left( \frac{1}{R} - \frac{1}{S} \right).$$

Thus the distance of the image is given, to the first order of small quantities, by the ordinary formula—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}.$$

When a broad pencil of light, with a focus P, is reflected or refracted at a spherical surface, the rays do not, after reflexion or refraction, all pass through the same point. If the pencil is made indefinitely narrow and along the axis, that is, the radius through P of the surface, the rays do converge, after reflexion or refraction, to a definite point Q on the axis. In the same way, a broad pencil with a point P on the axis of

a lens as focus will not, after refraction through the lens, have all its rays passing through a single point. But if the pencil is made indefinitely narrow and along the axis, it will have a definite focus, Q, after refraction.

The fact that, as the pencil is made broader, its rays pass further away from Q, the focus conjugate to P, is called **aberration** of the rays. It is due to the special shape of the surface at which the rays are deviated; and for spherical surfaces it is called **spherical aberration**.

The **aberration of a ray** proceeding from a point P on the axis is the distance of the point at which it cuts the axis after deviation from Q, the conjugate focus of P.

When the ray from P is taken indefinitely close to the axis, the point of its ultimate intersection with the axis after deviation is Q. So that such a ray has no aberration.

The **aberration of a pencil** is the greatest aberration of its rays.

We shall now consider the aberration of the rays, in certain cases, in small direct pencils; that is, the principal ray of the pencil will be supposed normal to the surface, and any other ray very nearly normal.

**Pencil refracted at Plane Surface.**—Let $\mu$ be the index of refraction. Let P N the normal from P on the surface $= u$. Let

FIG. 72.

the ray P R after refraction at R meet the surface in $q$, so that $Nq = v'$. And let $NR = y$. Then—

$$\sin RPN = \mu \sin RqN,$$
$$\therefore Rq = \mu \cdot RP;$$
$$i.e. \ \sqrt{v'^2 + y^2} = \mu \sqrt{u^2 + y^2}.$$

Thus approximately—

$$v'\left(1 + \frac{y^2}{2v'^2}\right) = \mu u \left(1 + \frac{y^2}{2u^2}\right);$$
$$v' = \mu u + \frac{y^2}{2}\left(\frac{\mu}{u} - \frac{1}{v'}\right).$$

Approximately $v' = \mu u$; and putting this value in the small term in $y^2$, we get—
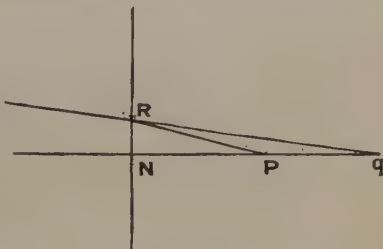
$$v' = \mu u + \frac{\mu^2 - 1}{\mu} \cdot \frac{y^2}{2u}.$$

This result is correct as far as the third power of $y$, that is, if we agree to neglect the fourth and higher powers. For if a more exact value of $v'$ were obtained, the next term would contain $y^4$.

The point to which the rays converge when $y$ is made indefinitely small is the focus conjugate to P ; and we see, as before, that it is given by—

$$v = \mu u.$$

The aberration of the ray R$q$ is—

$$v' - v = \frac{\mu^2 - 1}{\mu} \cdot \frac{y^2}{2u}.$$

The aberration is from or towards the surface according as $\mu$ is $>$ or $<$ 1.

This investigation and those which follow show more clearly how, as the ray approaches nearer to the axis, the point at which it intersects the axis after deviation approaches to a definite limiting position.

**To find where a Ray reflected at a Spherical Surface, and close to the Axis of the Surface, meets the Surface**
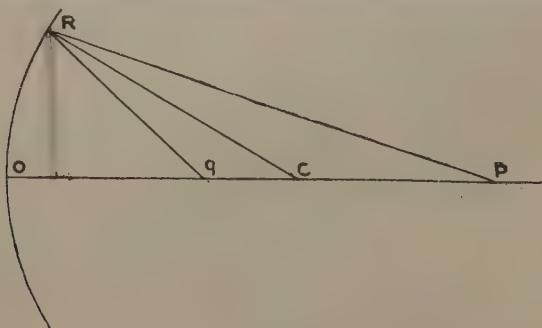


Fig. 73.

**after Reflexion.**—Let the ray P R after reflexion at R intersect the axis in $q$. Let CO = R. Let OP = $u$ ; O$q$ = $v'$. Let the perpendicular from R on OC = $y$.

Since RC bisects the angle PR$q$—

$$PR : Rq = P C : C q ;$$
$$\therefore PR . Cq = Rq . PC.$$

Now $PR^2 = PC^2 + CR^2 + 2PC \cdot CR \cos RCO$;

And $\sin RCO = \dfrac{y}{R}$.

$$\therefore \cos RCO = \sqrt{1 - \frac{y^2}{R^2}}$$
$$= 1 - \frac{y^2}{2R^2}, \text{ approximately.}$$
$$\therefore PR^2 = (u - R)^2 + R^2 + 2(u - R)R\left(1 - \frac{y^2}{2R^2}\right)$$
$$= u^2 - \frac{u - R}{R} \cdot y^2$$
$$= u^2\left\{1 - \left(\frac{1}{R} - \frac{1}{u}\right)\frac{y^2}{u}\right\};$$
$$\therefore PR = u\left\{1 - \left(\frac{1}{R} - \frac{1}{u}\right)\frac{y^2}{2u}\right\}, \text{ approximately.}$$

Similarly—

$$Rq = v'\left\{1 - \left(\frac{1}{R} - \frac{1}{v'}\right)\frac{y^2}{2v'}\right\}.$$

Thus we get—

$$(R - v')u\left\{1 - \left(\frac{1}{R} - \frac{1}{u}\right)\frac{y^2}{2u}\right\} = (u - R)v'\left\{1 - \left(\frac{1}{R} - \frac{1}{v'}\right)\frac{y^2}{2v'}\right\}.$$

Divide by $uv'R$, and we get—

$$\left(\frac{1}{v'} - \frac{1}{R}\right)\left\{1 - \left(\frac{1}{R} - \frac{1}{u}\right)\frac{y^2}{2u}\right\} = \left(\frac{1}{R} - \frac{1}{u}\right)\left\{1 - \left(\frac{1}{R} - \frac{1}{v'}\right)\frac{y^2}{2v'}\right\};$$
$$\therefore \frac{1}{v'} + \frac{1}{u} = \frac{2}{R} + \left(\frac{1}{R} - \frac{1}{u}\right)\left(\frac{1}{v'} - \frac{1}{R}\right)\left(\frac{1}{v'} + \frac{1}{u}\right)\frac{y^2}{2}.$$

Substituting in the last term the approximate value of $\dfrac{1}{v'}$ got by neglecting this term, we get—

$$\frac{1}{v'} + \frac{1}{u} = \frac{2}{R} + \left(\frac{1}{R} - \frac{1}{u}\right)^2 \cdot \frac{y^2}{R}.$$

If $v$ is the distance from O of Q, the focus conjugate to P—

$$\frac{1}{v} + \frac{1}{u} = \frac{2}{R};$$
$$\therefore \frac{1}{v'} - \frac{1}{v} = \left(\frac{1}{R} - \frac{1}{u}\right)^2 \cdot \frac{y^2}{R}.$$

$$v' - v = -\left(\frac{1}{R} - \frac{1}{u}\right)^2 \cdot \frac{y^2}{R} \cdot vv'$$

$$= -\left(\frac{1}{R} - \frac{1}{u}\right)^2 \cdot \frac{y^2}{R} \cdot v^2, \text{ nearly.}$$

**To find where a Ray refracted at a Spherical Surface, and close to the Axis of the Surface, meets the Axis after Refraction.**—Let the ray P R be refracted so
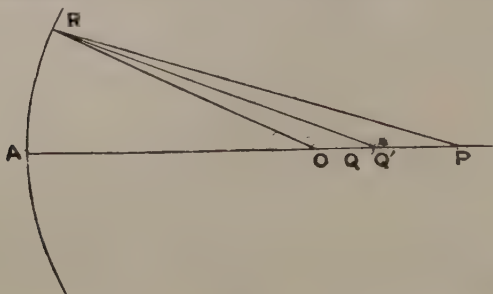


FIG. 74.

as to meet the axis P A in Q'. Let Q be the limiting position of Q' when R is indefinitely close to A. Let $AO = R$; $AP = u$; $AQ = v$; $AQ' = v'$. Let A R be a very small distance, and $= y$.

Then we have—

$$\mu OQ' \cdot RP = OP \cdot RQ'.$$

Now $RP^2 = R^2 + (u - R)^2 + 2R(u - R)\cos$

$$= R^2 + (u - R)^2 + 2R(u - R)\left(1 - \frac{y^2}{2R^2}\right), \text{ nearly,}$$

$$= u^2 - (u - R)\frac{y^2}{R}.$$

$$\therefore RP = u\left\{1 - \left(\frac{1}{R} - \frac{1}{u}\right)\frac{y^2}{2u}\right\}.$$

Similarly—

$$RQ' = v'\left\{1 - \left(\frac{1}{R} - \frac{1}{v'}\right)\frac{y^2}{2v'}\right\}.$$

Thus—

$$\mu(v' - R)u\left\{1 - \left(\frac{1}{R} - \frac{1}{u}\right)\frac{y^2}{2u}\right\} = (u - R)v'\left\{1 - \left(\frac{1}{R} - \frac{1}{v'}\right)\frac{y^2}{2v'}\right\}.$$

Dividing by $uv'R$, we get—

$$\mu\left(\frac{1}{R} - \frac{1}{v'}\right)\left\{1 - \left(\frac{1}{R} - \frac{1}{u}\right)\frac{y^2}{2u}\right\} = \left(\frac{1}{R} - \frac{1}{u}\right)\left\{1 - \left(\frac{1}{R} - \frac{1}{v'}\right)\frac{y^2}{2v'}\right\}.$$

$$\therefore \frac{\mu}{v'} - \frac{1}{u} = \frac{\mu - 1}{R} + \left(\frac{1}{R} - \frac{1}{v'}\right)\left(\frac{1}{R} - \frac{1}{u}\right)\left(\frac{1}{v'} - \frac{\mu}{u}\right)\frac{y^2}{2}.$$

Substitute for $v'$ in the small term containing $y^2$ from the approximate equation—

$$\frac{\mu}{v'} - \frac{1}{u} = \frac{\mu - 1}{R}.$$

Thus we get—

$$\frac{\mu}{v'} - \frac{1}{u} = \frac{\mu - 1}{R} + \frac{1}{\mu^2}\left(\frac{1}{R} - \frac{1}{u}\right)^2\left(\frac{1}{u} + \frac{\mu - 1}{R} - \frac{\mu^2}{u}\right)\frac{y^2}{2}.$$

$$= \frac{\mu - 1}{R} + \frac{\mu - 1}{\mu^2}\left(\frac{1}{R} - \frac{1}{u}\right)^2\left(\frac{1}{R} - \frac{\mu + 1}{u}\right)\frac{y^2}{2}.$$

**To find the Aberration of a Ray passing with Small Excentricity through a Thin Lens.**—Let P B A be the axis of the lens; R and S the radii of its front and back surfaces.

FIG. 75.

Let P R S T be the course of the ray. Let S R and T S produced meet the axis in $Q_1$, $Q_2$, at distances $v_1$, $v_2$ from the lens. Let A R, and therefore B S, nearly $= y$.

Then we have—

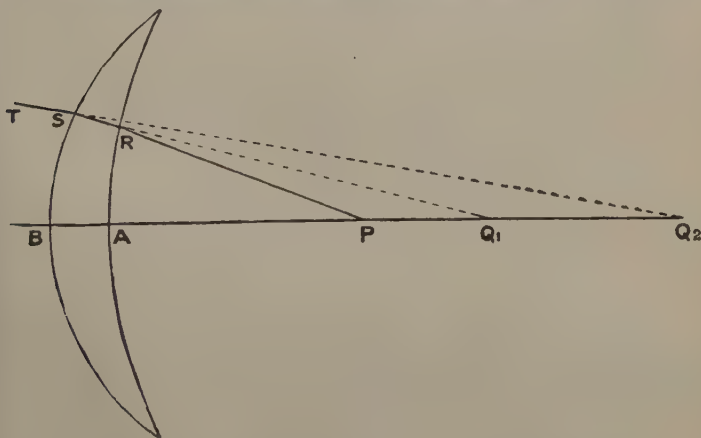$$\frac{\mu}{v_1} - \frac{1}{u} = \frac{\mu - 1}{R} + \frac{\mu - 1}{\mu^2}\left(\frac{1}{R} - \frac{1}{u}\right)^2\left(\frac{1}{R} - \frac{\mu + 1}{u}\right)\frac{y^2}{2}.$$

This equation connects $u$, $v_1$, and R. To find a similar equation connecting $v_2$, $v_1$, and S, notice that a ray along T S proceeding to $Q_2$, would, on refraction at the surface R S, proceed to $Q_1$. Thus in the above we have to replace $u$, $v_1$, and R by $-v_2$, $-v_1$, and $-$S; or we may change all signs and write $v_2$, $v_1$, and S. Thus we get—

$$\frac{\mu}{v_1} - \frac{1}{v_2} = \frac{\mu - 1}{S} + \frac{\mu - 1}{\mu^2}\left(\frac{1}{S} - \frac{1}{v_2}\right)^2\left(\frac{1}{S} - \frac{\mu + 1}{v_2}\right)\frac{y^2}{2}.$$

Subtracting, we get—

$$\frac{1}{v_2} - \frac{1}{u} = (\mu - 1)\left(\frac{1}{R} - \frac{1}{S}\right) + \frac{\mu - 1}{\mu^2}\left\{\left(\frac{1}{R} - \frac{1}{u}\right)^2\left(\frac{1}{R} - \frac{\mu + 1}{u}\right) - \left(\frac{1}{S} - \frac{1}{v_2}\right)^2\left(\frac{1}{S} - \frac{\mu + 1}{v_2}\right)\right\}\frac{y^2}{2}.$$

If $v$ is the limiting value of $v_2$, we have—

$$\frac{1}{v} - \frac{1}{u} = (\mu - 1)\left(\frac{1}{R} - \frac{1}{S}\right).$$

We may use the approximate value of $v_2$ got from this in the term containing $y^2$; and write—

$$\frac{1}{v_2} - \frac{1}{u} = (\mu - 1)\left(\frac{1}{R} - \frac{1}{S}\right) + \frac{\mu - 1}{\mu^2}\left\{\left(\frac{1}{R} - \frac{1}{u}\right)^2\left(\frac{1}{R} - \frac{\mu + 1}{u}\right) - \left(\frac{1}{S} - \frac{1}{v}\right)^2\left(\frac{1}{S} - \frac{\mu + 1}{v}\right)\right\}\frac{y^2}{2}.$$

The aberration of the ray ST is—

$$v_2 - v = -\frac{\mu - 1}{\mu^2}\left\{\left(\frac{1}{R} - \frac{1}{u}\right)^2\left(\frac{1}{R} - \frac{\mu + 1}{u}\right) - \left(\frac{1}{S} - \frac{1}{v}\right)^2\left(\frac{1}{S} - \frac{\mu + 1}{v}\right)\right\}\frac{v^2 y^2}{2}.$$

The aberration for a ray parallel to the axis is got by putting $u = \infty$; $v = f$; and is—

$$-\frac{\mu - 1}{\mu^2}\left\{\frac{1}{R^3} - \left(\frac{1}{S} - \frac{1}{f}\right)^2\left(\frac{1}{S} - \frac{\mu + 1}{f}\right)\right\}\frac{f^2 y^2}{2}.$$

R and S can be so chosen, subject to the condition that the lens shall have a given focal length, as to make the numerical value of this aberration a minimum.

If $\mu = \frac{3}{2}$, as is approximately the case for crown glass, the lens which will produce least aberration in a parallel pencil along the axis is such that $S = -6R$. Such a lens is called a *crossed lens*.

It would be possible so to choose the values of R and S for a lens of given focal length as to make the above expression, for the aberration of a ray from any point on the axis, vanish for certain values of $u$. The corresponding object-points are called **aplanatic foci** of the lens. Such a lens, it would be found, must have one surface convex and the other concave. An ordinary converging lens, with both surfaces convex, or with one plane, always produces a positive aberration ; and a diverging lens, with both surfaces concave, or with one plane, always produces a negative aberration. It will be useful to notice that what these conclusions come to is this : any ordinary lens will produce aberration in such a manner that its marginal portions will behave like a lens of numerically shorter focal length than its central portion.

Suppose we have a pencil R S $q$ reflected or refracted at a surface, so that Q $q$ is its aberration. Take two rays, T $q'$, U $q'$,



FIG. 76.

in the plane with R $q$, S $q$ intersecting the axis in $q'$, and meeting these rays in $h\,k$. Then $h\,k$ disappears when the rays T $q'$, U $q'$ lie on the axis, or when they coincide with R $q$, S $q$. So that for some intermediate position of T $q'$, U $q'$, $h\,k$ is a maximum. Let the figure represent this position. Then a circle on $h\,k$ as diameter, and at right angles to the axis, is the least area through which the entire pencil passes. This circle is called the **least circle of aberration**.

Suppose we take a series of normal sections to a very narrow astigmatic pencil, between its focal lines, the direction of these being at right angles to each other. As the section passes from the first to the second focal line, its breadth in the

direction of the first focal line diminishes to zero, and its breadth in the direction of the second focal line increases from zero. Thus in some position this section will have equal breadths in the directions of the two focal lines. This section is called **the circle of least confusion** of the pencil.

When an image of an object is seen by astigmatic pencils, we may regard the image as being the assemblage of all the circles of least confusion of the pencils.

The position and diameter of the circle of least confusion may be easily found. Let the orthogonal section of the pencil
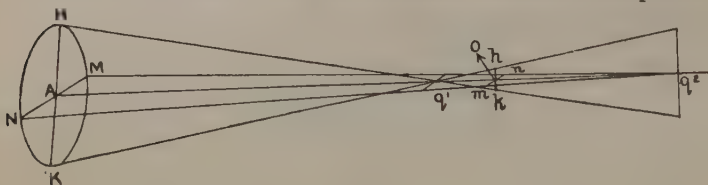
FIG. 77.

at A have breadths in the primary and secondary plane $HK = a$; $MN = b$. Let $Aq_1 = v_1$; $Aq_2 = v_2$. Let O be the centre of the circle of least confusion $h\,m\,k\,n$. And let $AO = x$. Then—

$$\frac{hk}{a} = \frac{x - v_1}{v_1}$$

$$\frac{mn}{b} = \frac{v_2 - x}{v_2}.$$

And $hk = mn$,

$$\therefore \frac{b}{a} = \frac{v_2(x - v_1)}{v_1(v_2 - x)},$$

$$\therefore x(av_2 + bv_1) = (a + b)v_1v_2,$$

$$\therefore x = \frac{(a + b)v_1v_2}{av_2 + bv_1}.$$

Again—

$$hk = \frac{a(x - v_1)}{v_1}$$

$$= \frac{ab(v_2 - v_1)}{av_2 + bv_1}.$$

Let us consider a broad pencil of light undergoing re-flexion or refraction at a spherical surface. Consider the reflected or refracted rays which lie close to one plane through the axis—the plane of the paper. These rays do not meet in

one point in the plane. But any assemblage of them making up a small astigmatic pencil, such as R S, will converge to a primary focal line, $q_1$, at right angles to the plane. Thus $q_1$ is the region of maximum concentration of the light in the pencil R S. The various little astigmatic pencils made up of the rays we are considering give rise to a series of primary focal lines, $q_1$, $q_1'$, $q_1''$, etc., arranged along a curve in the plane. This curve is



FIG. 78.

a region of maximum illumination or of maximum concentration of the rays of the reflected or refracted pencil; it is the locus of the intersections of the successive rays in the plane. The curve is called a **caustic curve.**

Considering the whole of the reflected or refracted pencil, we see that there is a surface which is a region of maximum illumination. This surface may be got by rotating the caustic curve about the axis. It is called a **caustic surface.**

A caustic by reflexion may be shown by placing a strip of bright metal bent into the form of a circular arc on a sheet of white paper, and so that a strong light, such as the light from the sun, may fall on the inner surface. The reflected light will give a caustic on the sheet of paper.

Another example of the caustic of a circle is the bright curve that may be seen on the top of a cup of tea by reflexion of light from the inside of the cup.

**Caustic formed by Parallel Rays reflected at Cylindrical Mirror.**—Suppose parallel light falls on a concave mirror in the form of a portion of a cylinder, the light being at right angles to the axis of the cylinder.

Let any ray, Q R, meet the surface at R (Fig. 79). Join R to O, the centre of the circular section of the cylinder through Q R. Bisect O R at T. Describe a circle with O T as radius, and another on T R as diameter, having centre O'. Let the latter circle be met by the reflected ray from R in P ; and draw O F parallel to Q R. Then—

$$\angle PO'T = 2\angle PRO' = 2\angle TOF.$$
$$\text{And } O'T = \tfrac{1}{2}OT ;$$
$$\therefore \text{arc } TP = \text{arc } TF.$$
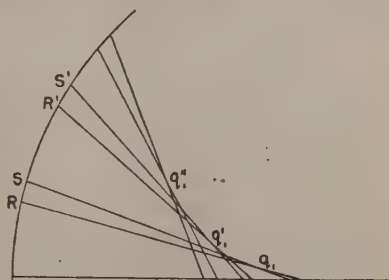
Thus if the circle R P T rolls on the circle T F, the point of R P T which is in one position of it in coincidence with F will trace out a curve on which P is a point. The line R P touches
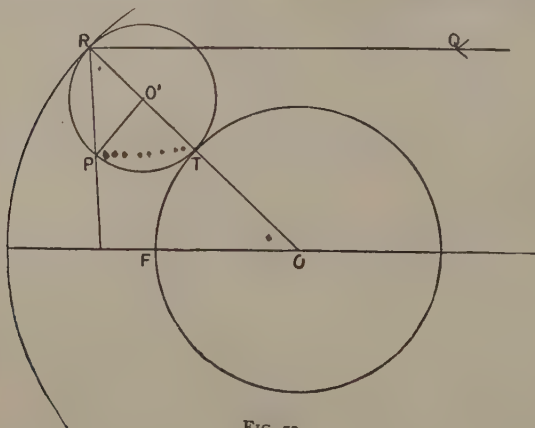


Fig. 79.
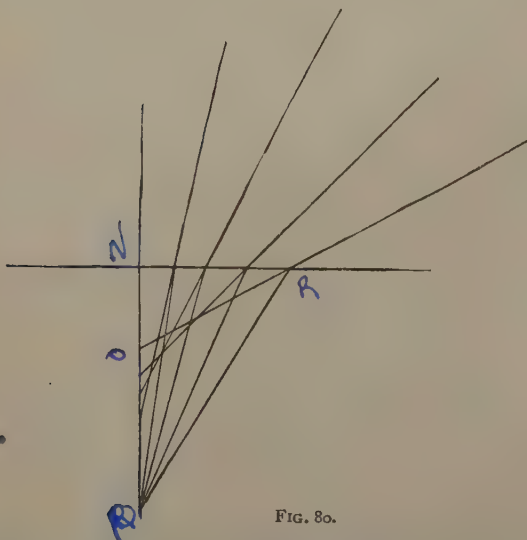


Fig. 80.

this curve at P. For at any instant the circle R P T is turning about its point of contact T with T F, and thus the direction of

motion of P, or of the curve through P, is at right angles to
T P, that is, along P R. This curve, therefore, is touched by
all the reflected rays such as R P  It is therefore the caustic
formed by these rays. The
curve is an *epicycloid*.

**Caustic, formed by
Refraction at a Surface,
of Rays coming from a
Point** ($\mu < 1$).—We have
in this case a virtual caustic.
We can find the geometrical
form of it by the following
device.

Let P R be an incident
ray. Draw P N normal to
the surface, and produce it,
making NS = PN. · Let the

Fig. 81.

refracted ray produced backward meet P S in O, and the
circle, through P, R, S, in L.
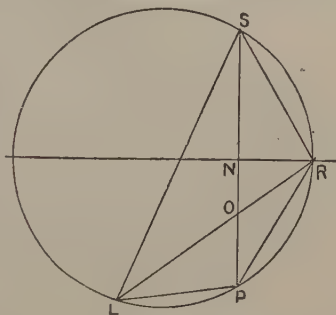
Then, since LO bisects the angle PLS—

$$\frac{LS}{LP} = \frac{SO}{OP};$$

$$\frac{SL + LP}{LP} = \frac{SP}{OP};$$

$$SL + LP = SP \cdot \frac{LP}{OP}$$

$$= SP \cdot \frac{\sin LOP}{\sin OLP}.$$

Now, LOP is equal to the angle of refraction at R ; and
OLP = RSP = RPS = angle of incidence.

$$\therefore SL + LP = \frac{2PN}{\mu}$$

Thus the locus of L is an ellipse ; and the refracted ray is
always a normal to this ellipse. The caustic is touched by all
the normals to the ellipse ; that is, it is the *evolute* of the
ellipse.

## CHAPTER V.

### *COMPOSITE CHARACTER OF LIGHT.*

**Newton's Experiment.**—Newton showed, by the following experiment, that the light from the sun, or, as a rule, from any source of illumination, is not simple in character, but composite, consisting of various sorts of simple light, which differ from each other in colour and in their degrees of refrangibility by a given refracting substance.

Sunlight is allowed to enter a darkened room through a small hole, O, in a shutter, and to fall on a white screen.
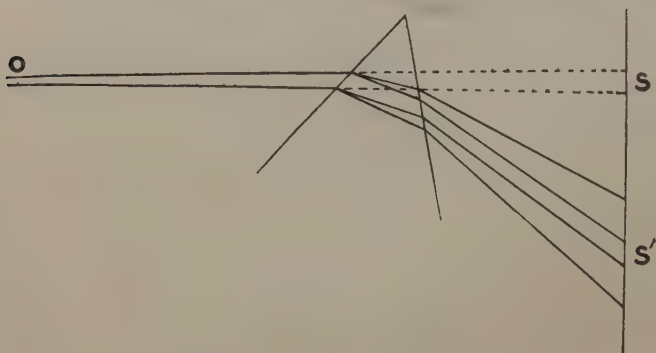


FIG. 82.

The light may be direct from the sun, or reflected by a mirror. An image of the sun would be formed on the screen at S, as in a camera obscura. Now a prism is interposed in the path of the rays, so as to deviate them and cause the image to be formed at another part of the screen, at S'.

It is noticed about this image—

(1) That it is considerably elongated in the direction at right angles to the edge of the prism.

(2) That it is brilliantly coloured with all the colours of the rainbow, from red in the least deviated portion to violet in the most deviated.

The colours through which the image passes from red to violet are innumerable in variety of tints. Newton considered them to be seven in number, namely, *red, orange, yellow, green, blue, indigo, violet.*

The variegated image thus produced by the sun's light is called a *solar spectrum*.

If a coloured screen is used to receive the image, the spectrum will not be complete, some of the colours appearing in their proper places, that is, where they would be on a white screen, but the rest of the screen being unilluminated, or, at the most, comparatively dark. The screen will be illuminated in that portion only of the spectrum which corresponds to its own colour. Thus a screen of pure red will only show illumination in the red part of the spectrum. The colour of a screen may, however, not be the same as any one of the colours of the spectrum. In that case it is made up of more than one, and it will show illumination in more than one part of the spectrum, or, in an extended part. This is the case with the great majority of substances, there being few substances in nature whose colours are simple.

A substance will, then, be illuminated by light of its own colour only; or of such colours as its colour contains, if its colour is composite. And we infer that a white screen contains all the colours in the solar spectrum, that is, that white is made up of all these colours. Further, since these are all the simple colours that can be obtained from the sun's light, it follows that the sun's light is composed of light of all the colours that go to make up white, and in the right proportions, or is *white light*.

In the spectrum formed as described above, since the light passes by imperceptible variations through the various degrees of refrangibility, and since the light of any definite colour would form an image of finite dimensions, it follows that the images of various colours encroach upon or overlap each other. Thus the illumination at any point of the spectrum is not produced by light of one simple kind, but by lights of all degrees of refrangibility between certain limits. The spectrum so produced is said to be an impure one.

**Pure Spectrum.**—A pure spectrum is one in which the light at each point is simple or of a definite refrangibility. To obtain a pure spectrum, the aperture used to admit the light must be a very narrow slit, A (Fig. 83); there is generally some arrangement for adjusting the width of it. A lens, L, is used to produce an image, B, of the slit on a screen. A prism, P, is placed between the lens and the screen, with its edge parallel to the slit, and rotated till the deviated image of A is in the position of minimum deviation. Since there is not very much difference in the deviations produced in the images formed by the various

kinds of light, they will all be practically in their positions of minimum deviation for the same position of the prism. The prism now forms at C, with the light of any simple kind, a real image of the slit, at the same distance from itself as B was.
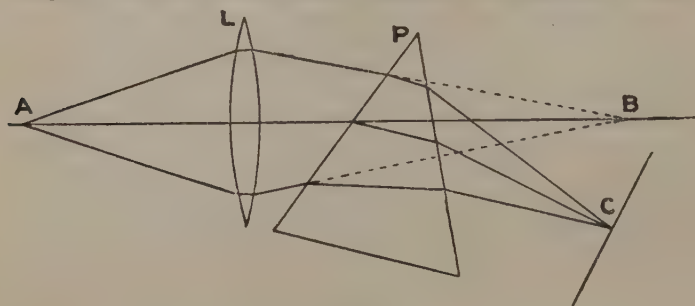


FIG. 83.

The screen is then placed to receive the spectrum produced at C, the assemblage of all the deviated images of the slit, its distance from the prism being made the same as the distance of B from the prism.

On account of the finite width of the slit, there will still be more or less overlapping of the images at C. But the slit can be made extremely narrow, so that the spectrum obtained will be, for all practical purposes, a pure one.

The light at any point of the pure spectrum, being quite simple or of a definite refrangibility, is called *homogeneous*, or *monochromatic*. The composition of light cannot be inferred from its colour, for two lights precisely alike in appearance may each be composed of monochromatic constituents such that those in the one light are entirely different from those in the other; and a monochromatic colour may be exactly matched by a combination of other colours all quite different from it.

A pure spectrum may also be obtained by letting the light from the slit fall first on the prism set in the position of minimum deviation, and then letting it fall on a lens which will produce an image of the slit formed by the light of each separate colour; a screen being set to receive these images which constitute the spectrum. The diagram (Fig. 84) shows this arrangement.

The simplest way, however, to see a pure spectrum is to let the eye itself bring to a focus the rays of each colour proceeding from the slit and refracted through the prism.

That is, in the above diagram, the lens of the eye takes the place of the lens, and the retina takes the place of the screen. Let the slit be at the distance of most distinct vision from the
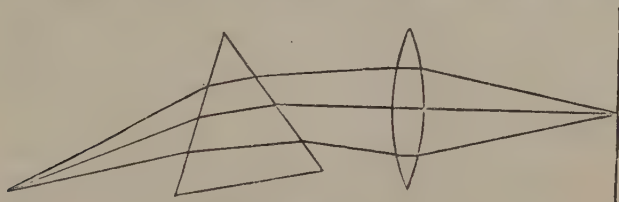
eye, the prism placed before the eye with its edge parallel to the slit. Turn the prism till the image, or series of coloured images, of the slit is at the position of minimum deviation. These images are now also at the distance of most distinct vision. The eye, being focussed on them, will see a pure spectrum.

When a composite ray of light is refracted, the constituent rays are deviated by different amounts. This is called **dispersion** of the light; and the angle between two given rays after deviation is called **the dispersion of the two rays.**

Let $\mu_r$, $\mu_v$, $\mu$ be the indices of refraction of a given medium for the extreme red and violet rays, and for the rays of mean refrangibility. Let $D_r$, $D_v$, $D$ be the minimum deviations which would be produced in these rays by a prism of the substance of very small refracting angle, $i$. Then—

$$D_r = (\mu_r - 1)i,$$
$$D_v = (\mu_v - 1)i,$$
$$D = (\mu - 1)i.$$

Then the limiting ratio of dispersion to deviation is $\dfrac{D_r - D_v}{D}$, and this is equal to $\dfrac{\mu_r - \mu_v}{\mu - 1}$.

This fraction is called the **dispersive power** of the substance.

**Recomposition of White Light.**—If the colours into which white light has been broken up by dispersion be recombined, white light will again be produced. This can be done in several ways.

1. A prism exactly like that used to produce dispersion may be placed to receive the light just as it leaves the first, having its faces parallel to those of the first, and its refracting edge

turned the other way. Then every ray will be deviated by this prism just as much as it was by the first, and in the opposite sense. Thus any composite ray is dispersed by the first prism, and the rays into which it is broken up are brought into parallelism by the second. Suppose a pencil, A B C D, to pass through the two prisms, and the emerging light to fall on a screen at *a b c d*. Take the two extremes,
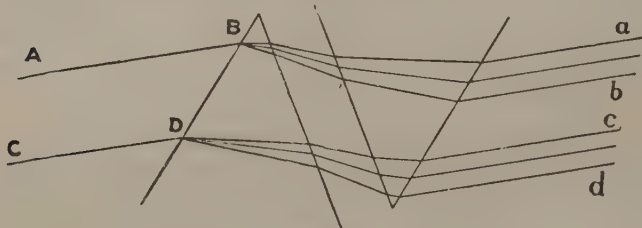


Fig. 85.

A B, C D, of the pencil. The ray A B gives rise to a series of rays falling on the screen from *a* to *b*, and giving the light which varies from red to violet as we pass from *a* to *b*. The ray C D gives rise, in the same way, to light on the screen varying from red to violet as we pass from *c* to *d*. And similarly for intermediate rays. All the portion from *b* to *c* will be illuminated with rays of all colours, and will appear white. In the boundary portion *a b*, the colours of the spectrum successively disappear as we pass from *b* to *a*, the violet being first wanting just beyond *b*, then as we go on towards *a* the blue *as well as* the violet, and so on, till at *a* the extreme red only remains. This edge of the illuminated area is thus coloured with a reddish tinge. In the same way in the other edge, *c d*, the colours of the spectrum successively disappear as we pass from *c* to *d*, the red going first. The edge is coloured a bluish green.

2. If the light from the dispersing prism is received on a convex lens or a concave mirror, and then brought to a focus on a screen placed so that it and the prism are at conjugate foci, the colours will be recombined on the screen, and give rise to white light, at any rate in the middle of the portion they illuminate.

3. If a cardboard disc be coloured in sectors with the colours of the spectrum, and then spun rapidly about its centre, the appearance will be whitish, on account of the persistence of the visual impressions produced by the colours.

**Dispersion produced by Lens.**—Suppose a small object to be illuminated by white light; and let a lens be used to form an image of the object. If the light proceeding from the object were of one simple colour, the substance of the lens would have a definite refractive index for it, and the image would be formed in a definite position. But the indices of refraction are different for the different colours; the rays of various colours will be variously deviated; and a series of images, of various colours, will be formed, at slightly different distances from the lens.

This property of the lens, whereby the rays do not converge to a definite focus, but are spread out according to their colours, even when only a small portion of the lens round the axis is used, is called **chromatic aberration.**

The diagram shows the way in which the rays of various colours, which have fallen on a lens from a luminous or visible
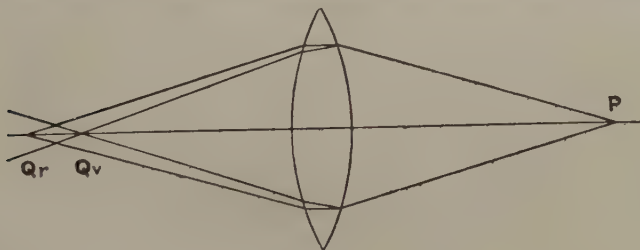


Fig. 86.

point, P, proceed on leaving the lens, giving rise to a series of coloured conjugate foci, or images of P, ranged along the axis from $Q_r$ the red, to $Q_v$ the violet, image; $Q_v$ being nearest to the lens for the case drawn, because the violet rays are most refrangible.

This effect may be shown in the following way: Hold a white screen so as to receive the image of P. When the screen is held at $Q_v$, an image of P will be formed on it of the violet rays. The other sets of rays will not yet have come to a focus, but will be spread out, and especially the red rays. The image formed of P will have an edge that is badly defined, and coloured with a reddish tinge. When the screen is held at $Q_r$, an image is formed of the red rays; the other sets of rays have all passed their foci, and will be spreading out, especially the violet. The image will have a badly defined edge coloured with a bluish-green tinge. The best image will

be formed somewhere between $Q_{,,}$ and $Q_{,,,}$ where there is least chromatic aberration of the rays.

Chromatic aberration is a serious difficulty in the use of lenses, and we must consider how this difficulty has to be overcome. Chromatic aberration is produced in a lens in this way: An excentrical ray which falls on the lens passes into and out of the lens at points where the surfaces are not parallel. The ray is thus deviated by the portion of the lens through which it passes, in just the same way as it would be deviated by a prism. And it undergoes dispersion just as it would on passing through a prism. If the rays could be deviated without dispersion, that is, so as all to be brought to the same focus, there would be no chromatic aberration. The problem, then, is to produce deviation by refraction without dispersion. It will be simpler, first, to consider how this may be done with prisms.

Suppose a prism of a given sort of glass to produce dispersion in a parallel pencil of light, the prism being set to produce minimum deviation. Let the light, after passing through this prism, fall on a second one set with its edge parallel to that of the first, and turned to produce deviation in the opposite direction, and so as also to produce minimum deviation of the light which falls upon it. Let the second prism be of the same material as the first. Then if it is made so as just to cut out the dispersion produced by the first, or to bring the rays into parallelism again, it must have the same refracting angle as the first. And, having the same refracting angle, it will also just cut out the deviation produced by the first prism. In this way, then, we can only destroy the dispersion at the expense of all the deviation. It remains to be seen whether with prisms of different substances we can do otherwise.

Newton inferred, from the limited number of experiments which he made, that it was impossible to combine two prisms to produce deviation without dispersion; that is, that whenever a beam of light passed through two prisms so as to emerge colourless, it also emerged parallel to its old direction. This would mean, for prisms of small angles, that the dispersion is always proportional to the deviation, whatever the substance of the prism; or that the dispersive powers of all substances are the same. We know now that this conclusion is not correct for all substances. If, for example, we use a crown-glass prism and a flint-glass prism, and they are made to produce equal deviations, the flint will produce more dispersion

for two given rays; or, if they are made to produce equal dispersions, the crown will produce more deviation. In flint glass, in fact, the ratio of dispersion to deviation is greater than in crown, or flint glass has a greater dispersive power than crown. A prism of crown and a prism of flint glass could, then, be made and arranged so that the flint would cut out all the dispersion produced by the crown, but leave in some of the deviation. In this way is solved the problem of producing deviation in a pencil of light, by means of prisms, without producing dispersion.

In what has been said we must, in strictness, be supposed to refer to the dispersion of *two* rays, say the extreme red and violet. For if these two rays be brought together in such a combination of prisms as that just described, it does not follow that the other rays will be combined with them. The dispersions between the various pairs of rays do not bear quite the same ratios in the two substances. Thus when the red and the violet are brought together, another ray, say the mean yellow, will not be brought quite into coincidence with them. The prisms producing the same dispersion between red and violet do not produce quite the same dispersion between red and yellow. Supposing, then, that the light came from a slit parallel to the edges of the prisms, there will be formed on the screen an image of the slit which is slightly coloured, the colours on one side consisting of a combination of colours from the two ends of the spectrum, and those on the other of a combination of colours from the middle. The appearance thus produced on the screen is called a **secondary spectrum.** This is much less coloured and spread out than the primary spectrum.

The fact which gives rise to secondary spectra, that the dispersions between the various pairs of rays in different refracting substances are not proportional, is called **irrationality of dispersion.**

We may use prisms of three different substances, so as to recombine three different rays of the spectrum, say red, yellow, and violet. The appearance then produced on the screen would present very little colour indeed. It would be a **tertiary spectrum.**

With a combination of prisms, say of two, we may use a very narrow slit and a lens to get the spectrum produced by white light on a white screen, just as was done with a single prism. We should then have resulting a pure secondary spectrum. We should have the images formed by red and

violet lights superposed (these being the colours brought together); and side by side with these the images resulting from other combinations of pairs of colours, the whole being spread out only a very little way over the screen. As the slit is made broader, the image will become white in the middle, the edges only being coloured, one of them with colours from the two ends of the primary spectrum, the other with colours from the middle.

A combination of prisms or lenses which produces a colourless image is said to be an **achromatic combination**.

We have next to consider the question of making a combination of lenses which shall be achromatic. Now, such a combination may be required to be achromatic in one of two distinct ways. We shall, in all cases, begin by supposing that the lenses have a common axis.

1. The combination may be made achromatic for images formed by *centrical* pencils. Then the images of a visible point, formed by such small pencils of the various colours, must all be superposed at another point. If the lenses are separated by appreciable distances, that the pencils may be centrical for all, it is clear that point and image must be on the common axis. If the lenses are of inconsiderable thickness and in contact, the same oblique pencil can be approximately centrical for all. In this case points close to, but not on, the axis would also have formed of them achromatic images by means of centrical pencils. A combination of lenses to be used to form an image on a screen requires to be made achromatic in this sense; so also do the object-glasses of microscopes and of ordinary telescopes, for it is centrical pencils that are used in all these cases.

2. The combination may be made *achromatic for excentrical pencils*, that is, so that a ray of light passing excentrically through the lenses may have its various constituents emerging parallel, and therefore, practically, in coincidence. This is necessary for a combination of lenses which is to be looked through; for an eye sees the various points of an object through various parts of the lens, and thus by excentrical pencils. Compound eye-pieces of microscopes and telescopes have to be made achromatic in this sense.

We shall now consider how a lens may be made which will form a real image of an object, and bring together the red and the violet rays by which it is formed.

Suppose we have a point, P, illuminated by white light, on the axis of a convex lens of crown glass. The rays from P along

P R will undergo dispersion, the red being least deviated, and the violet most; so that the red ray will take the course P R S $Q_r$, and the violet the course P R T $Q_v$. Now, behind the crown lens let a concave one of flint glass be placed, having the same axis as the first. The rays S $Q_r$, T $Q_v$ will be deviated away from the axis, and the latter will be the more deviated; so that they may, by a suitable combination of lenses, be brought to meet the axis in the same point Q. For a combination of two lenses to produce a focus which is achromatic for two colours, conjugate to a point P on the axis, its focal length must be the same for both colours, and thus it will produce achromatic conjugate foci for all points on the axis.

Now, suppose the focal length of the lenses so chosen as to produce an achromatic image, Q, of the point P on the axis. Then if P is taken at the same distance as before from the common centre, but now a little off the axis, its images $Q_r$, $Q_v$, formed by the first lens, will be at the same distances from the centre as before, for the distance of image and object are connected by the same formula—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

whether the line joining them is the axis or is slightly inclined to the axis. And these points, $Q_r$, $Q_v$, will give images on the line joining them to the centre, all together, and at the same distance as before. Thus in this case, too, the image of P is achromatic. And the combination will form an achromatic image of a small object on the axis.
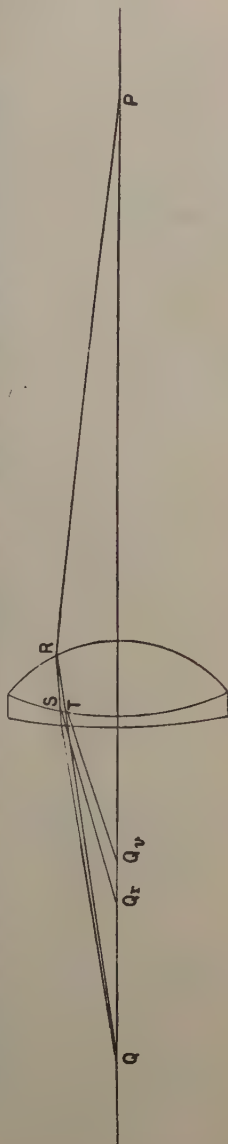
It should be noticed that this combination, having a centre and a focus, may be treated as a single lens for purposes of graphic construction, determination of magnification, and so on.

**Condition of Achromatism for a Combination of Two Thin Lenses for Pencils passing centrically and close to the Axis.**—Suppose two lenses combined as just described, their thicknesses being inconsiderable. To be achromatic for two colours, their combined focal lengths for each of the colours must be the same. Let the two colours have refractive indices $\mu_1$, $\mu_1'$ in the first lens; and $\mu_2$, $\mu_2'$ in the second. Let the four radii of curvature of the lenses be $r_1$, $s_1$; $r_2$, $s_2$. Let the focal lengths of the first lens for the two colours be $f_1$, $f_1'$; and of the second $f_2$, $f_2'$. Let the focal lengths of the combination for the two colours be F, F'. Then—

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$$
$$= (\mu_1 - 1)\left(\frac{1}{r_1} - \frac{1}{s_1}\right) + (\mu_2 - 1)\left(\frac{1}{r_2} - \frac{1}{s_2}\right);$$
$$\frac{1}{F'} = \frac{1}{f_1'} + \frac{1}{f_2'}$$
$$= (\mu_1' - 1)\left(\frac{1}{r_1} - \frac{1}{s_1}\right) + (\mu_2' - 1)\left(\frac{1}{r_2} - \frac{1}{s_2}\right).$$

But $F = F'$.

$$\therefore 0 = (\mu_1' - \mu_1)\left(\frac{1}{r_1} - \frac{1}{s_1}\right) + (\mu_2' - \mu_2)\left(\frac{1}{r_2} - \frac{1}{s_2}\right);$$
$$\therefore \frac{\mu_1' - \mu_1}{\mu_1 - 1} \cdot \frac{1}{f_1} + \frac{\mu_2' - \mu_2}{\mu_2 - 1} \cdot \frac{1}{f_2} = 0.$$

Or, by the differential calculus, since focal length, F, of the combination is given by—

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2},$$

the differential of this, in passing to another refractive index, must vanish.

$$\therefore d \cdot \frac{1}{f_1} + d \cdot \frac{1}{f_2} = 0.$$
$$\text{But } \frac{1}{f_1} = (\mu_1 - 1)\left(\frac{1}{r_1} - \frac{1}{s_1}\right),$$

$$\therefore d \cdot \frac{1}{f_1} = d\mu_1 \left( \frac{1}{r_1} - \frac{1}{s_1} \right) = \frac{d\mu_1}{\mu_1 - 1} \cdot \frac{1}{f_1}.$$

$$\text{So } d \cdot \frac{1}{f_2} = \frac{d\mu_2}{\mu_2 - 1} \cdot \frac{1}{f_2}.$$

$$\therefore \frac{d\mu_1}{\mu_1 - 1} \cdot \frac{1}{f_1} + \frac{d\mu_2}{\mu_2 - 1} \cdot \frac{1}{f_2} = 0.$$

If we require to find what lenses will make a combination achromatic for two given colours, and having a given focal length, F, for them, this condition and the equation—

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}.$$

are sufficient to give us $f_1$ and $f_2$.

The above condition of achromatism is that the sum of the products for the two lenses of dispersive powers and reciprocals of focal lengths shall be zero.

For a similar combination of three lenses to be achromatic for two given colours, we would get a similar condition in three terms. Having the three focal lengths at disposal, we can make them satisfy another condition, so that we can make the combination achromatic for one of the two given colours and a third. So that with three lenses of different substances we can make a combination achromatic for three colours.

In general, with any number of thin lenses combined as above, the condition for achromatism for two colours may be written—

$$\Sigma \left( \frac{d\mu}{\mu - 1} \cdot \frac{1}{f} \right) = 0.$$

Suppose there are $n$ lenses; then, if we choose $n$ colours by taking them two and two, we can get $n - 1$ *independent* equations, of which the above is a type. These, with the equation among the $f$'s for the focal length of the combination, are $n$ equations, from which to find the $n f$'s. Thus, with $n$ lenses we can recombine $n$ colours.

The number of independent conditions among the $f$'s is limited as follows: Suppose there are, for example, three colours, red, yellow, and violet, to be combined. To combine the red with the violet introduces one condition, and to combine the red with the yellow another. To combine the yellow with the violet introduces no new condition; this is done by combining the red with each of them. In the same way, $n$ colours will be combined by combining one with each

of the other $n - 1$; thus giving $n - 1$ independent conditions or equations.

Suppose we have two lenses of focal lengths $f_1, f_2$, and refractive indices $\mu_1, \mu_2$, for a given colour. Let them be separated by an interval $a$. To find the condition that they should be achromatic for a parallel pencil along the axis.

The distance from the second lens at which the light is brought to a focus is given by—

$$\frac{1}{v} = \frac{1}{f_1 + a} + \frac{1}{f_2}.$$

Now let us pass to another colour, so that the $f$'s and the $\mu$'s undergo small changes, $df_1, df_2, d\mu_1, d\mu_2$. The corresponding variation of $v$ must be zero.

We have already seen that—

$$d \cdot \frac{1}{f_1} = \frac{d\mu_1}{\mu_1 - 1} \cdot \frac{1}{f_1}.$$

We may write—

$$\frac{1}{v} = \frac{\dfrac{1}{f_1}}{1 - \dfrac{a}{f_1}} + \frac{1}{f_2}.$$

And since the differential of this must vanish, we must have, by elementary differential calculus—

$$0 = \frac{\left(1 + \dfrac{a}{f_1}\right) d \cdot \dfrac{1}{f_1} - \dfrac{1}{f_1} \cdot a \cdot d \cdot \dfrac{1}{f_1}}{\left(1 + \dfrac{a}{f_1}\right)^2} + d \cdot \frac{1}{f_2};$$

$$\frac{d \cdot \dfrac{1}{f_1}}{\left(1 + \dfrac{a}{f_1}\right)^2} + d \cdot \frac{1}{f_2} = 0;$$

$$\frac{\dfrac{d\mu_1}{\mu_1 - 1} \cdot \dfrac{1}{f_1}}{\left(1 + \dfrac{a}{f_1}\right)^2} + \frac{d\mu_2}{\mu_2 - 1} \cdot \frac{1}{f_2} = 0;$$

$$\frac{f_1}{(a + f_1)^2} \cdot \frac{d\mu_1}{\mu_1 - 1} + \frac{1}{f_2} \cdot \frac{d\mu_2}{\mu_2 - 1} = 0.$$

This result could be obtained by algebra, by supposing

$\mu_1, \mu_2$ to vary slightly and become $\mu_1', \mu_2'$; but the process would be much longer.

**The Chromatic Aberration** between the rays of two colours in a small pencil along the axis of a lens, after refraction through the lens, is the distance between the foci for these two colours.

The chromatic aberration of the pencil is the aberration between its extreme colours.

*To find the chromatic aberration between two colours in a small axial pencil refracted through a lens.*

Let $u$ be the distance of the focus P of the pencil; $v_r, v_v$ those of the conjugate foci for the red and violet rays; $f_r, f_v$ the corresponding focal lengths of the lens; and $\mu_r, \mu_v$ the refractive indices. The chromatic aberration is $\pm (v_r - v_v)$.

Now
$$\frac{1}{v_r} - \frac{1}{u} = \frac{1}{f_r};$$
$$\frac{1}{v_v} - \frac{1}{u} = \frac{1}{f_v}.$$

Therefore, subtracting—
$$\frac{v_r - v_v}{v_r v_v} = \frac{1}{f_v} - \frac{1}{f_r}$$
$$= \frac{\mu_v - \mu_r}{\mu - 1} \cdot \frac{1}{f}.$$

$f$ and $\mu$ being mean focal length and refractive index.

Now, $v_r$ and $v_v$ are very nearly equal, so that we may write $v_r v_v = v^2$, where $v$ is the distance of the focus conjugate to P for the mean rays. Thus—
$$v_r - v_v = \frac{\mu_v - \mu_r}{\mu - 1} \cdot \frac{v^2}{f}.$$

Or, by the differential calculus—
$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f};$$
$$\therefore -\frac{dv}{v^2} = d \cdot \frac{1}{f};$$
$$dv = -\frac{d\mu}{\mu - 1} \cdot \frac{v^2}{f}.$$

in which $dv$ and $d\mu$ stand for $v_v - v_r$ and $\mu_v - \mu_r$.

Suppose a combination has to be made **achromatic for excentrical pencils.** Then the rays of various colours from any point must enter the eye in coincidence. For this

condition to be fulfilled, the images, of various colours, of any small object formed by the last lens must have their corresponding points in straight lines with the eye. Now, supposing the pencils by which the eye sees the image to be inclined at small angles to the axis, and since the various coloured images are not far from each other, this condition is sufficiently attained by making these images all of the same size.

The combination of a thin convex and a thin concave lens in contact, which is achromatic for centrical pencils, is also achromatic for excentrical pencils. For the coloured images formed by both lenses of a small object are all in one place, and thus at the same distance from the common centre of the lenses. Thus they are all of the same size.

Consider two lenses, focal lengths for mean rays $f_1, f_2,$ separated by a distance $a$. The condition that this combination should be achromatic for excentrical pencils coming from a small object on the axis will involve the position of the object. We shall consider merely the case in which the object is at a great distance; so that the combination may be achromatic for rays parallel to the axis; and we shall suppose the lenses made of the same material.

The first lens forms an image practically at its principal focus. This is at a distance, $a + f_1$, from the centre of the second lens. Suppose an image is formed in the second lens at a distance, $v$, from its centre. Then—

$$\frac{1}{v} = \frac{1}{f_2} + \frac{1}{a + f_1}.$$

Now, if $b$ is the distance of the object from the first lens, the full magnification of the image is—

$$\frac{f_1}{b} \cdot \frac{v}{a + f_1}.$$

This must be constant for all the images.

$$\therefore \frac{1}{f_1} \cdot \frac{a + f_1}{v} \text{ must be constant.}$$

$$\text{Now } \frac{1}{f_1} \cdot \frac{a + f_1}{v} = \frac{1}{f_1} \left( 1 + \frac{a + f_1}{f_2} \right)$$

$$= \frac{1}{f_1} + \frac{1}{f_2} + \frac{a}{f_1 f_2}.$$

This must be constant for variations of $f_1$ and $f_2$ caused by variation in $\mu$.

$$\therefore d \cdot \frac{1}{f_1} + d \cdot \frac{1}{f_2} + a \left( \frac{1}{f_1} \cdot d \cdot \frac{1}{f_2} + \frac{1}{f_2} \cdot d \cdot \frac{1}{f_2} \right) = 0.$$

Therefore, dropping the factor $\dfrac{d\mu}{\mu - 1}$ throughout, we get—

$$\frac{1}{f_1} + \frac{1}{f_2} + a \left( \frac{1}{f_1 f_2} + \frac{1}{f_2 f_1} \right) = 0;$$

$$a = -\frac{f_1 + f_2}{2};$$

which is the required condition.

<div align="center">EXAMPLE.</div>

A compound achromatic lens of focal length 40 cms. is to be constructed of two thin crown-glass and flint-glass lenses in contact, the surfaces that are in contact having a common radius of 25 cms. The optical characters of the glasses employed being as follows, namely :—

|  |  |  |  | Dispersive power. |  | Refractive index for middle of spectrum. |
|---|---|---|---|---|---|---|
| Crown glass | ... | ... | ... | 0·21 | ... | 1·5 |
| Flint glass | ... | ... | ... | 0·45 | ... | 1·6 |

calculate the radius of the second face of each lens, and establish the formulæ employed in the calculation. (Lond. B.Sc. Hons., 1884.)

<div align="center">

# CHAPTER VI.

## *DEFECTS IN IMAGES FORMED BY LENSES AND MIRRORS.*

</div>

**Defects in the Images formed by Lenses.**—We have already considered the defects of colouring in the image formed by a lens, or chromatism; we shall here consider only the defects of form.

We have seen that a lens will only produce a true geometrical image of a point if the point is on its axis; but a very close approximation is produced, and generally sufficient for practical purposes, to a true geometrical image, of a small object through which the axis of the lens passes. We shall now consider a little more fully the shape of this image, and how far it resembles the object.

We have seen that a small object on the axis at a distance $u$ gives rise to an image at a distance $v$; $v$ being determined by the formula—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}.$$

Also that if the object be in one plane perpendicular to the axis, the image will, too, be in such a plane, and similar to the object; the ratio of the distance between two points of the image to that between the corresponding points of the object being $\frac{v}{u}$.

Now, suppose all the points of the object are not in the same plane perpendicular to the axis. Let the differences of
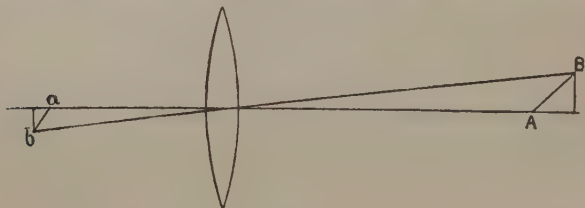


FIG. 88.

the $u$'s for two points be $du$. This is very nearly the distance between the corresponding planes. Let the corresponding difference in the $v$'s be $dv$.

Then since—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f},$$

we have—

$$\frac{dv}{v^2} - \frac{du}{u^2} = 0;$$

$$\therefore \frac{dv}{du} = \frac{v^2}{u^2}.$$

Thus the ratio of the size of the image to that of the object, when measured parallel to the axis, is not, as a rule, the same as when it is measured perpendicular to the axis.

The image is then *distorted* in this respect.

**Curvature of Image.**—Consider the image formed by centrical pencils by a lens, of a small object on the axis all in one plane perpendicular to the axis; say of a small straight line. We have seen that, to the second order of small quantities, the focal lines of any point of the object agree with each other and with the image, and this is given by the ordinary formula—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}.$$

To investigate the curvature of the image, however, we must go to a closer degree of approximation.
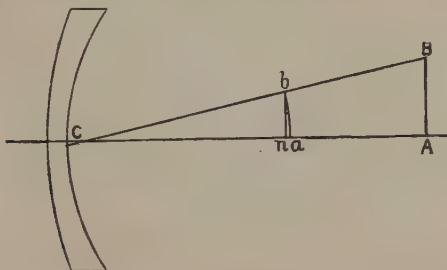


FIG. 89.

Suppose the small straight line A B (A being on the axis) to give the image $ab$. Let CB = $u$.

Now, B forms two focal lines in the neighbourhood of $b$, $b$ being generally taken as the circle of least confusion between these two.

Let the distances of the focal lines from C be $v_1$, $v_2$.

Then we have—

$$\frac{1}{v_1} - \frac{1}{u} = \frac{\mu \cos r - \cos i}{\cos^2 i}\left(\frac{1}{R} - \frac{1}{S}\right),$$

$$\frac{1}{v_2} - \frac{1}{u} = (\mu \cos r - \cos i)\left(\frac{1}{R} - \frac{1}{S}\right).$$

$r$ and $i$ being both small, we may write, correct to the *third* order of small quantities,—

$$\cos r = 1 - \frac{r^2}{2},$$

$$\cos i = 1 - \frac{i^2}{2},$$

$$(\cos i)^{-2} = 1 + i^2,$$

$$i = \mu r.$$

Thus—

$$\frac{\mu \cos r - \cos i}{\cos^2 i} = \left(\mu - \frac{\mu r^2}{2} - 1 + \frac{i^2}{2}\right)(1 + i^2)$$

$$= \left(\mu - 1 - \frac{i^2}{2\mu} + \frac{i^2}{2}\right)(1 + i^2)$$

$$= \mu - 1 + i^2\mu - \frac{i^2}{2} - \frac{i^2}{2\mu} \quad \text{(correct to}$$

the third order of small quantities)

$$= \mu - 1 + \frac{i^2}{2\mu}(2\mu^2 - \mu - 1)$$

$$= (\mu - 1)\{1 + \frac{i^2}{2\mu}(2\mu +)\}.$$

And—

$$\mu \cos r - \cos i = \mu - 1 - \frac{\mu r^2}{2} + \frac{i^2}{2}$$

$$= \mu - 1 - \frac{i^2}{2\mu} + \frac{i^2}{2}$$

$$= (\mu - 1)\left(1 + \frac{i^2}{2\mu}\right).$$

Thus—

$$\frac{1}{v_1} - \frac{1}{u} = \frac{1}{f}\{1 + \frac{i^2}{2\mu}(2\mu + 1)\},$$

$$\frac{1}{v_2} - \frac{1}{u} = \frac{1}{f}\left(1 + \frac{i^2}{2\mu}\right).$$

The distance, $v$, of the image $b$ is, therefore, given by—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}(1 + ki^2),$$

where $k$ is some positive constant, whether we take the circle of least confusion or one of the focal lines for the image.

Draw $b\,n$ perpendicular to the axis. We have—

$$\frac{1}{Ca} - \frac{1}{CA} = \frac{1}{f}.$$

Multiply this equation by $\cos i$ (*i.e.* approximately by $1 - \frac{i^2}{2}$), and subtract from the above.

Then, since $u \cos i = CA$, we have—

$$\frac{1}{v} - \frac{\cos i}{Ca} = \frac{1}{f}\left(1 + ki^2 - 1 + \frac{i^2}{2}\right),$$

$$\frac{na}{v \cdot Ca} = \frac{(2k + 1)i^2}{2f}.$$

In the limit $i^2 v \cdot Ca = bn^2$.

And if $\rho$ is the radius of curvature of the curve $a\,b$ at $a$, measured from $a$ leftwards—

$$\frac{1}{2\rho} = \text{limit of } \frac{na}{bn^2};$$

$$\therefore \frac{1}{\rho} = \frac{2k + 1}{f}.$$

This *distortion of curvature* would be produced by any lens, even when only centrical pencils are used, these being allowed to pass only through a small portion of the lens near the axis, and all others intercepted, so as to avoid spherical aberration. It is experienced in the image produced by the object-glass of a telescope, this image being rendered convex towards the observer.

Figs. 90, 91, and 92 show the curvatures produced in the three cases; with real image formed by convex lens,
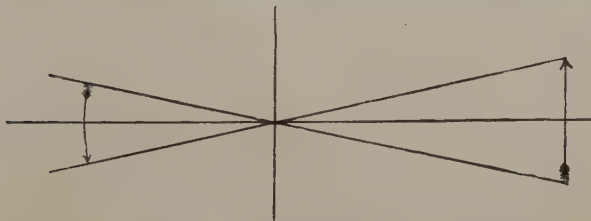


FIG. 90.

virtual image formed by convex lens, and image (virtual) formed by concave lens. In the first two cases the curvature is to the right, or towards the positive direction, towards the
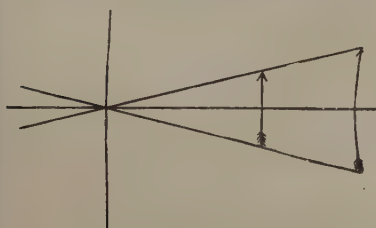


FIG. 91.                     FIG. 92.

incident light; in the third it is in the opposite direction, as the formula given above for $\rho$ would show in all cases.

In any case the curvature may be most easily explained in the following way: Any lens with spherical surfaces will behave towards oblique centrical pencils as a lens of numerically smaller focal length would do if the pencils were direct. This will have the effect in a convex lens of throwing the virtual image further off, and drawing the real image nearer up; and in a concave lens of drawing the image (virtual) nearer up.

**Linear Distortion** is the distortion in which the ratio

of the distances of two points of the image from the axis is
not the same as the ratio of the corresponding points of the
object; that is, when points far from the axis appear to be
spread out too much or drawn in too much as compared with
those near to the axis.

**Angular Distortion** is the distortion in which the angle
between the distances of two points of the image from the
axis is not the same as the corresponding angle for the object.

When an image is formed of centrical pencils, as the
image that may be formed on a screen by a convex lens of
which only a small portion near the axis is used, there is
neither linear nor angular distortion.

Let us consider the distortion in the image of the object
formed by a lens, seen by an eye on the axis of the lens.
In this case the image seen will be formed by *excentrical*
pencils, the rays coming to the eye through various points
of the lens.

The linear distortion produced by a lens may be most
easily explained by reference to its spherical aberration.

Suppose an eye placed at E on the axis of a convex lens
to view the virtual image of an object A B C. Let us consider
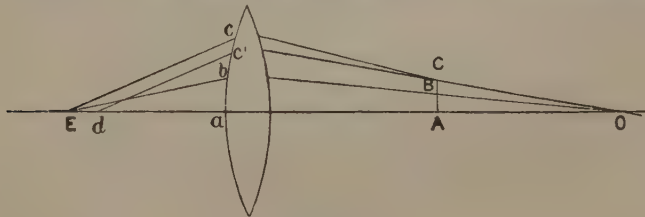


FIG. 93.

the paths of the rays that reach the eye from the various points
of the object. Let O be the conjugate focus to E. O must
be further from the lens than A is, because A is nearer than the
principal focus, and O is further off. All rays which go to E
would, on being produced backward, go through O but for
spherical aberration. The ray B *b*, along O B produced, where
B is a point close to the axis, goes to E. The ray C *c'*, along
O C produced, meets the axis at a point, *d*, nearer to the lens
than E. The ray from C which *does* reach E diverges more
widely from the axis than C *c'* does. Let it be C *c* E. Thus,
in the image which E sees of A B C, the images of B and
C are on the productions of E *b* and E *c*. If they were
along E *b* and E *c'*, A B and A C would appear magnified

proportionately. But, as it is, A C is more magnified than A B is. The parts of the object further away from the axis appear to be spread out too much from the axis, and there is linear distortion.

Similarly, the opposite sort of distortion produced in a real image by a convex lens may be explained as Fig. 94 shows.
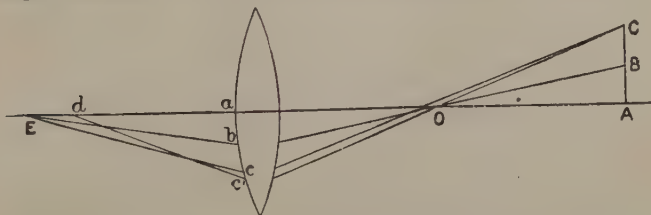


FIG. 94.

In this case O is nearer to the lens than A, because the conjugate focus to A is between E and the lens. The ray from C which reaches E meets the lens at a point, *c*, nearer to the axis than *c'* is. The images of B and C are seen along E *b*, E *c*. Thus the image of C is relatively nearer to the axis than that of B is.

Again, suppose an eye at E to view an object, A B C, through a concave lens. Let O be the focus conjugate to E.
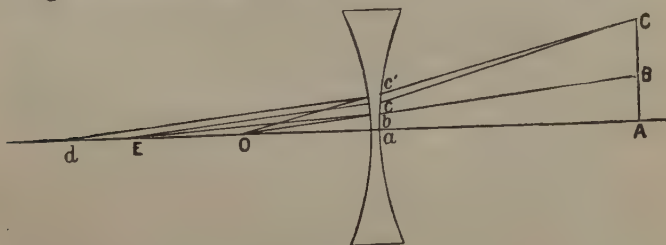


FIG. 95.

B being a point on the object very near to the axis, the ray B *b*, going to O, is deviated to E. The ray C *c'*, going to O, is deviated to *d*, a point further off than E. The ray from C, which *does* reach E, meets the lens at *c*, a point nearer to the axis than *c'* is. Thus in the image which E sees of A B C, the images of B and C are on E *b*, E *c* produced. Thus the image of C is relatively nearer to the axis than that of B is.

The linear distortion produced in the three cases may be illustrated as follows: Suppose, in each case, that the object

viewed is a square, the axis of the lens passing through its centre perpendicular to its plane. Then, in the first case—
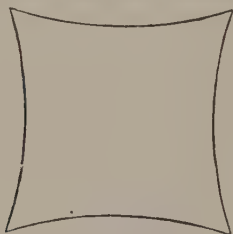


FIG. 96.

FIG. 97.

virtual image in convex lens—the image will have the appearance of Fig. 96; but in either of the other cases it will have the appearance of Fig. 97.

**Curvature of Image formed by Excentrical Pencils.** —The image, formed by excentrical pencils, of an object in a plane at right angles to the axis, that is, the image seen by an eye which views the object through the lens, will have curvature of the same sort in each case as that which exists in an image formed by centrical pencils, and already shown in Figs. 90, 91, 92. The explanation in this case is similar. Any lens with spherical surfaces acts, on account of spherical aberration, in its marginal portions as a lens of numerically smaller focal length than its proper focal length for direct centrical pencils. It was to this distortion that the well-known appearance due to linear distortion used to be attributed. Hence the latter distortion is generally called by its old name, want of **flatness of field.**

This distortion (of curvature in the image formed by excentrical pencils) is not of great importance; but it exists. The similar distortion of the image formed by centrical pencils is of much more importance; for it produces an effect on the image of a plane object formed on a screen; so that when the image of the central portion is formed distinctly, that of the marginal portion is not, and *vice versâ*. In the image which appears to the eye on looking through the lens it does not matter much if some parts are a little nearer than they ought to be, provided the parts keep their angular distances from the axis in the proper proportions; that is, provided there is no *linear* distortion.

**Defects in the Images formed by Spherical Mirrors.** —The defects of distortion in the images produced by spherical

mirrors will be similar to those in the images produced by lenses.

If the object is not all in one plane perpendicular to the axis, its dimensions perpendicular and parallel to the axis will not, in general, be magnified proportionately, the magnification at right angles to the axis being $\frac{v}{u}$, and parallel to the axis $\frac{v^2}{u^2}$.

If we have an object all in a plane at right angles to the axis of a spherical mirror, the area of the mirror being very limited, then of the pencils of light from the various points of the object, all, except that from the point on the axis, will, on reflexion, give rise to astigmatic pencils, producing more or less indistinct images of the points. Taking the circles of least confusion of these pencils as the image, the image of the plane object will be curved, as in the case of a lens.

Let us next consider linear distortion in a spherical mirror.

Suppose a concave spherical mirror produces a real image, $a\,b\,c$, of the object A B C, which image is seen by an eye at E
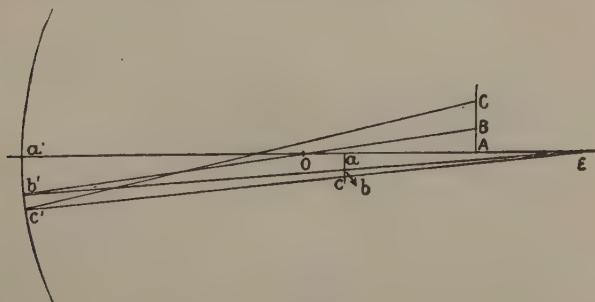


FIG. 98.

on the axis. Take O the conjugate focus to E. B, $b$ being near the axis, the pencil from $b$, which reaches E, passed, before reflexion, through O. If there were no spherical aberration, the pencil from the point C, far from the axis, would also have come through O, and thus we should have $ac : ab = a'c' : a'b' = $ AC : AB. But the pencil from $c$ passed, before reflexion, through a point on the axis nearer to the mirror than O; thus we have $ac : ab < $ AC : AB. And the image is contracted in the marginal parts.

Suppose we have a virtual image of A B C formed in a concave mirror. The rays from $c$ to E, before reflexion, came as

if from a point nearer the mirror than O ; $ac : ab > AC : AB$
The image is expanded in the marginal parts.
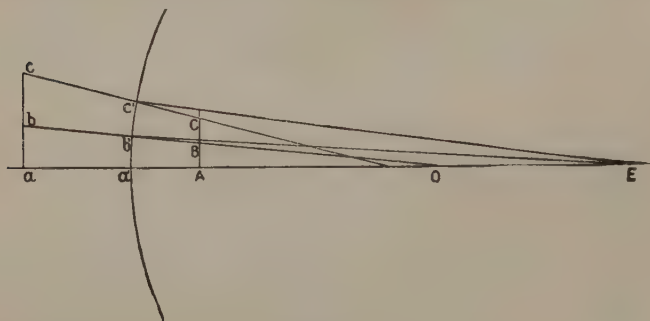


FIG. 99.

Suppose we have an image (virtual) of A B C formed in a
convex mirror.  The rays from $c$ to E, before reflexion, were
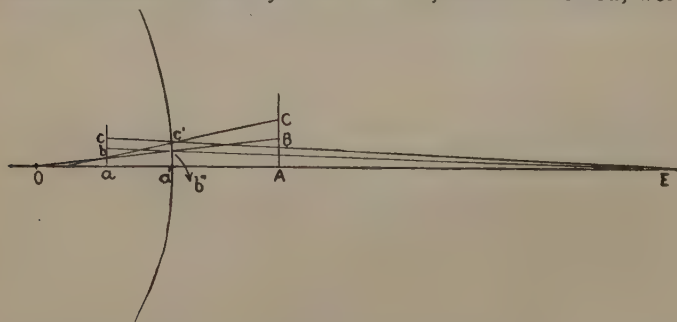


FIG. 100.

proceeding to a point nearer to the mirror than O.  Thus
$ac : ab < AC : AB$.  The image is contracted in the marginal
parts.

Notice that we have here the same results as for the corre-
sponding cases with lenses, these being, respectively, real and
virtual image with convex lens, and image (virtual) with
concave lens.

## CHAPTER VII.

### *THICK LENSES.*

WE shall now consider the position of the image, formed by means of a thick lens, of a small object on the axis of the lens. Let the symbols have the following meanings :—

$\mu$ = refractive index of lens.

$t$ = thickness of lens.

$r$ = radius of front surface.

$s$ = radius of back surface.

$u$ = distance of object from front surface.

$v$ = distance of image from back surface.

$v'$ = distance of image formed by refraction at front surface from that surface.

Then we have, for the refractions at the two surfaces, the equations—

$$\frac{\mu}{v'} - \frac{1}{u} = \frac{\mu - 1}{r} \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{1}{v} - \frac{\mu}{v' + t} = \frac{1 - \mu}{s} \quad . \quad . \quad . \quad (2)$$

From these we must eliminate $v'$.

Clearing of fractions, we may write the equations—

$$\mu u r - r v' - (\mu - 1) u v' = 0.$$
$$(v' + t)s - \mu v s + (\mu - 1)(v' + t)v = 0.$$

From these we get, by equating the two values they give for $v'$,—

$$\frac{\mu u r}{r + (\mu - 1)u} = \frac{\mu v s - t\{s + (\mu - 1)v\}}{s + (\mu - 1)v}.$$

Multiplying up, and rearranging, we get—

$$(\mu - 1)\{\mu(r - s) + (\mu - 1)t\}uv + s\{\mu r + (\mu - 1)t\}u$$
$$- r\{\mu s - (\mu - 1)t\}v + trs = 0.$$

We can throw this into the form—

$$\frac{1}{v + \beta} - \frac{1}{u + \alpha} = \frac{1}{f}.$$

For this equation may be written—

$$uv - (f - \beta)u + (f + \alpha)v + f(\beta - \alpha) + \alpha\beta = 0.$$

Equating coefficients, we get—

$$\frac{1}{(\mu - 1)\{\mu(r - s) + (\mu - 1)t\}} = \frac{-f + \beta}{\mu rs + (\mu - 1)st}$$
$$= \frac{-f - \alpha}{\mu rs - (\mu - 1)rt} = \frac{\alpha\beta}{trs}$$

These equations are satisfied by writing—

$$f = -\frac{\mu rs}{(\mu - 1)\{\mu(r - s) + (\mu - 1)t\}};$$
$$\alpha = \frac{rt}{\mu(r - s) + (\mu - 1)t};$$
$$\beta = \frac{st}{\mu(r - s) + (\mu - 1)t}.$$

We see that the positions of object and image can be expressed by precisely the same formula as that used for a thin lens if $u$ and $v$ are measured, not from the surfaces, but from the points whose distances from the front and back surfaces are $\alpha$ and $\beta$. These points are called respectively the **first** and **second principal points** of the lens. The planes through them perpendicular to the axis are called the **principal planes**.

**Magnification produced by Thick Lens.**—We shall consider the magnifications produced at the surfaces. When a small object gives an image by refraction at a single spherical surface, corresponding points of object and image lie on the same radii. Thus the linear dimensions of object and image will be in the ratio of their distances from the centre. If, then, an object at distance $u$ from the surface of radius $r$, refractive index $\mu$, gives an image at distance $v'$, the magnification is—

$$m = \frac{v' - r}{u - r}.$$
$$\text{But } \frac{\mu}{v'} - \frac{1}{\mu} = \frac{\mu - 1}{r}.$$
$$\therefore \frac{\mu(v' - r)}{v'r} = \frac{u - r}{ur}.$$
$$\therefore m = \frac{v'}{\mu u}.$$

In the case of the thick lens this would be the magnification

produced at the first surface, and that produced at the second surface would be—

$$\frac{\mu v}{v'}.$$

Thus the magnification produced by the lens is—

$$\frac{v}{u}.$$

Let H, H' be the principal points of a thick lens; P and Q a pair of conjugate foci. Let a pair of corresponding rays through P and Q meet the principal planes in the points R
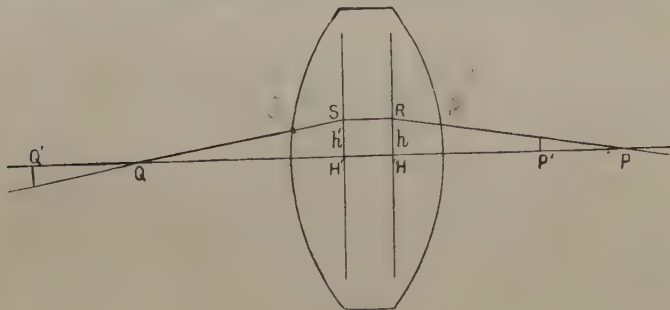


FIG. 101

and S, at distances $h$ and $h'$ from the axis. We can find the relation between $h$ and $h'$ by the following device: A small object, P', will give rise to an image at a known point, Q'. Now, we know the relation of size of image to size of object: so that if we take PR as a ray proceeding from a point of the object, we know how the corresponding ray, S Q, must be drawn, for we know through what point of the image to draw it.

Let the distances of P, P' from H be $u$, $u'$; and those of Q, Q' from H' be $v$, $v'$. Let $\beta$, $\beta'$ be the lengths of object and image. Then—

$$\frac{h}{u} = \frac{\beta}{u - u''}$$

$$\frac{h'}{v} = \frac{\beta'}{v - v'}.$$

And—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f} = \frac{1}{v'} - \frac{1}{u'};$$

$$\therefore \; \frac{u - u'}{uu'} = \frac{v - v'}{vv'} \; ;$$

$$\frac{h}{h'} = \frac{\beta}{\beta'} \cdot \frac{u}{v} \cdot \frac{v - v'}{u - u'}$$

$$= \frac{\beta}{\beta'} \cdot \frac{v'}{u'} = 1.$$

$$h = h'.$$

That is, corresponding rays, before entry and at emergence, meet the principal planes in points equidistant from the axis.

This gives a useful method of drawing the paths of corresponding rays. Suppose a ray through the point P on the axis to cut the axis again at Q after passing through the lens. Draw PR to meet the first principal plane in R. Draw RS parallel to the axis to meet the second principal plane in S. Then SQ is the path of the emergent ray.

Suppose an object in the first principal plane. This may be a real object if the plane is outside the lens; or may be a virtual object formed by converging rays in any case. The image will be in the second principal plane. For if in the equation—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

we make $u = 0$, then $v = 0$. By what we have just seen, the rays through any point of the object diverge on emerging as if they came from a point at the same distance from the axis. Thus the image is of the same size as the object. The principal planes are called, in consequence, **planes of unit magnification.**

We may also prove these properties of the principal planes as follows: $u$ and $v$ being distances of image and object, measured from the principal points, the magnification is $\frac{v}{u}$.

$$\text{But since } \frac{1}{v} - \frac{1}{u} = \frac{1}{f},$$

$$\frac{v}{u} = 1 - \frac{v}{f}.$$

Therefore when $u$ and $v$ vanish, the magnification becomes unity.

It follows that a ray going before entry to any point on the first principal plane, proceeds on emergence from a point at an equal distance from the axis on the second principal plane. That is, the line R S, in the above figure, is parallel to the axis.

It should be noticed that the principal points are a pair of conjugate foci, since when $u = 0$, $v = 0$.

A ray proceeding to the first principal point emerges from the second parallel to its original direction. For if P$p$, Q$q$ are object and image, P$p$ : Q$q$ = HP : H'Q. Thus $p$ H and H'$q$ are parallel.



FIG. 102.
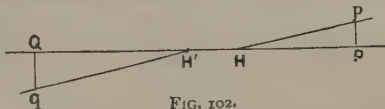
From what we have seen, it follows that graphic constructions are to be made in the same way as for thin lenses, only with the addition of the space between the principal planes. If this space were removed, the points H, H' would coincide; and so would the points R and S in the last figure.

If the media on the two sides of a lens are any whatever, not necessarily the same, a pair of conjugate foci can be found such that if a ray proceeds towards one before entering the lens, it emerges from the other in a direction parallel to its original direction. These points are called **nodal points**. When the lens is situated in air, the nodal points coincide with the principal points.

The formula for the focal length has been given above; the principal focus is at this distance from the second principal point; and just as for a thin lens, there is another principal focus, for rays coming in the opposite direction. This is at a distance which is numerically the same from the first principal focus.

Except so far as it involves $t$, the thickness, the focal length will be just the same as for a thin lens of the same substance and the same curvatures. When $t$ is small, the focal length will, as a rule, be positive for a lens thinnest in the middle, and negative for a lens thickest in the middle. But it may happen that $t$ may decide the sign of $f$. For suppose $r - s$ negative, and numerically less than $\dfrac{t(\mu - 1)}{\mu}$. The focal length of the lens is then of opposite sign to that of a thin lens of the same substance and curvatures.

Suppose a lens has equal and opposite curvatures, so that $r = -s$; has small thickness; and is made of crown glass,

so that $\mu = \frac{3}{2}$ nearly. Then we have, approximately, from the values of $a$ and $\beta$—

$$a = \frac{rt}{\frac{3}{2} \cdot 2r} = \frac{t}{3}; \quad \beta = -\frac{t}{3}.$$

Thus the principal points are the points of trisection of the portion of the axis intercepted by the surfaces.

Suppose two thick lenses, of focal lengths $f_1, f_2$, are placed with their axes coincident, and so that their principal points, when taken in order, are $a_1, \beta_1; \ a_2, \beta_2$. Let $a_2 \beta_1 = x$.

Let an object at a distance $u$ from $a$ give an image by the first lens at a distance $v'$ from $\beta_1$; and then an image by the second lens at a distance $v$ from $\beta_2$. Then—

$$\frac{1}{v'} - \frac{1}{u} = \frac{1}{f_1};$$

$$\frac{1}{v} - \frac{1}{v' + x} = \frac{1}{f_2}.$$

Eliminating $v'$, we find—

$$(f_1 + f_2 + x)uv + (f_1 f_2 + f_1 x)v - (f_1 f_2 + f_2 x)u - f_1 f_2 x = 0.$$

Now, if the combination is equivalent to a single lens of focal length F, and principal points at distances $y$ and $z$ from $a_1$ and $\beta_2$, we have—

$$\frac{1}{v - z} - \frac{1}{u - y} = \frac{1}{F}.$$

Or—

$$uv + (F - y)v - (F + z)u + F(y - z) + z = 0.$$

Equating coefficients, we have—

$$\frac{1}{f_1 + f_2 + x} = \frac{F - y}{f_1(f_2 + x)} = \frac{F + z}{f_2(f_1 + x)} = \frac{F(z - y) - yz}{f_1 f_2 x}.$$

These equations are satisfied by writing—

$$F = \frac{f_1 f_2}{f_1 + f_2 + x}; \quad y = -\frac{x f_1}{f_1 + f_2 + x}; \quad z = \frac{x f_2}{f_1 + f_2 + x}.$$

Thus a single lens can be found optically equivalent, for direct centrical pencils, to the given combination of two.

Again, it follows that if we have three coaxal lenses, two of these can be replaced, as above, by a single lens; and then this can be combined with the third, and replaced by a single one.

And in the same way any number of coaxal lenses can be replaced by a single lens with definite principal points and principal foci, as far as the action on small axial pencils is concerned. This result is important in the theory of optical instruments.

## CHAPTER VIII.

### *EYE-PIECES—MICROSCOPES—TELESCOPES—OTHER OPTICAL APPARATUS.*

**Vision through a Lens.**—A single convex lens is some-times used as a *magnifying-glass*. It is sometimes arranged on a stand in a manner suitable for viewing small objects. This arrangement is called a **simple microscope**. Suppose the lens has focal length $f$; if object and image are at distances $u$ and $v$ from the lens, we have—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f};$$

$$\therefore m = \frac{v}{u} = 1 - \frac{v}{f}.$$

Suppose the lens used to produce an erect, virtual image. Then $v$ is positive; and $f$ is negative. Thus the image is larger the further off it is. But the image is required to sub-tend a large angle at the eye rather than to be large itself. We shall consider, therefore, how this result is to be obtained.

Let the **least distance of distinct vision** be denoted by D. This is the least distance at which the object can be held from the eye so that it may be seen distinctly, or so that the lens of the eye can produce of it a true image on the retina. Or, in practice, rather it means the least distance at which the object may be placed so as to be seen distinctly without over-straining the eye. For a normal eye this distance is generally taken to be about 10 inches, or 25 centimetres. For this distance or for any greater, a normal eye can see an object distinctly and easily, but not at any less distance.

If, then, an image of given size is formed at distance D from the eye, it subtends the greatest possible visual angle at the eye. If the eye is held close up to the lens (the distance between them being negligible), for the greatest visual angle the image must be formed at distance D from the lens; for as its distance is increased its size is increased, but in a smaller ratio. And if the eye is held at a distance from the lens, to bring the image to a distance, D, from the eye, it must be brought nearer to the lens, and so its size diminished. Thus to get the greatest visual angle at the eye, the eye must be held close to the lens, and the image formed at a distance, D, from either. The visual angle thus obtained, supposing object and image

K

to be very small, is *size of image* ÷ D. The greatest visual angle that the object can subtend at the eye for distinct vision is *size of object* ÷ D. The ratio of these is called the **magnifying power** of the lens, and is—

$$1 - \frac{D}{f}.$$

The same result may easily be obtained algebraically. Suppose the eye at a distance $x$ from the lens. Then, since size of image = size of object $\times \left( 1 - \dfrac{v}{f} \right)$, and distance of image from eye = $x + v$; we want to have $\dfrac{1 - \dfrac{v}{f}}{x + v}$ as large as possible.

Put $-f = f'$ (a positive quantity).

$$\therefore \frac{f' + v}{x + v} \text{ must be as large as possible.}$$

Let $x + v = y$, the distance of the image from the eye, and this expression becomes—

$$\frac{f' + y - x}{y},$$

Or—

$$\frac{f'}{y} + 1 - \frac{x}{y}.$$

In this $x$ can have any value, and $y$ any value not less than D. It is clear, then, that for *any* value whatever of $y$, this is greatest when $x = 0$; and then it is greatest when $y$ is as small as possible, or $=$ D.

**Eye-pieces.**—If a single lens is used to look at an object, or at an image that has been formed by an optical system, we have seen that the image formed is defective: it is distorted, the marginal portions undergoing too much or too little displacement from the axis, on account of the spherical aberration produced by the lens. The field of view thus produced is said to want *flatness*. It used to be supposed that this appearance of the image was due to curvature, so that the image of an object in a plane at right angles to the axis is not formed entirely in another such plane. This effect of curvature *is* produced; but the appearance of linear distortion is not due to it, but is something independent (see pp. 118–120).

To remedy the defect of linear distortion in the image, instead of using a single lens to view the image with, in a

microscope or telescope, a combination of lenses is used. For if, instead of producing the required deviation of a given excentrical ray or pencil by means of a single lens, we cause the deviation to be produced by several lenses separated from each other, the spherical aberration will be diminished. Such a combination of lenses is called an eye-piece. The lens next the eye is called the eye-lens; and the one furthest from the eye the field-lens. Let us examine, by the help of a diagram, the way in which spherical aberration may be diminished by
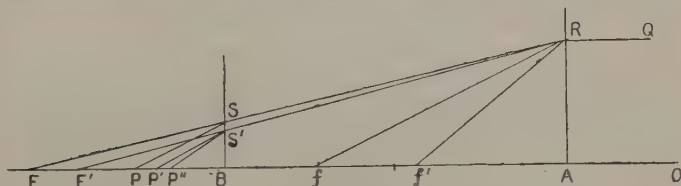


FIG. 103.

means of two lenses. Suppose we have an assemblage of rays parallel to a given line O A B (this may be the optical axis of an instrument), and we require to act on these rays in such a manner that the ray Q R receives a definite deviation. This may be done by means of a single lens at A. This lens must have a definite focal length. Let $f$ be its principal focus; so that the rays parallel to Q R and indefinitely close to O A pass, after deviation by the lens, through $f$; and, if the lens were aplanatic, that is, producing no spherical aberration, Q R and all the rays would do so too. Suppose the lens placed at A, with principal focus $f$, produces the aberration $f f'$ in the ray Q R, so that this ray passes through $f'$. Now, if instead of this lens we place at A a lens of the same shape, but having a greater focal length, A F, its radii of curvature must be increased in the ratio A$f$ : A F, and, the distance A R remaining the same, the aberration, F F', must be diminished in the ratio A F : A$f$, as is obvious from the formula for the aberration of a ray parallel to the axis.

[If, for instance, AF $= 2$A$f$, so that, A R being very small, the deviation produced in the ray Q R by the lens originally at A is half that produced by the lens by which it is replaced, then FF' $= \frac{1}{2}ff'$.] Now suppose another lens placed at B so as to further deviate the ray Q R, and to give it finally the full deviation required, that is, to bring it into parallelism with R $f$. We can with these two lenses, at A and B, produce an aberration less than $f f'$.

The lens at A deviates the ray Q R to R S′ F′, instead of to R S F, the position produced by an aplanatic lens at A of equal focal length. The lens at B, if aplanatic, would turn the ray R S to P, and R S′ to P′. Now, applying to this lens the formula—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f},$$

if $d\,v$, $d\,u$ are corresponding small variations in $v$ and $u$, we have $dv = du \cdot \dfrac{v^2}{u^2}$. Thus PP′ = FF′. $\dfrac{\text{BP}^2}{\text{BF}^2}$, an aberration in the final result due to the aplanatism of the lens at A. [Supposing, as above, that the lens at A produces half the full deviation in the ray Q R, then $\angle$ SFB = $\frac{1}{2}\angle$ SPB, whence BP = $\frac{1}{2}$BF, and PP′ = $\frac{1}{4}$FF′.] The lens at B further produces aberration on its own account, which may be represented by P′ P″; but this may be made small by reducing the distance B S, that is, by reducing the focal length, and suitably adjusting the position of the lens at B. Thus the final aberration P P″ may be made considerably smaller than $ff'$. By increasing A F, the aberration F F′ is diminished proportionately. And, to produce the requisite deviation of the ray, as A F is increased, the ratio BP : BF is diminished, and so PP′ : FF′ is diminished. By putting the lens B close up to F, the aberration P′P″ is diminished. Thus we may, by suitably choosing the lenses and the distance A B, theoretically diminish the aberration to any extent, always producing the required deviation in a given ray, Q R. By suitably arranging the curvatures of the lens-surfaces, the aberration may be still further diminished. It is found to be unnecessary to make the distance between the lenses very great. Thus eye-pieces can be made practically aplanatic, and at the same time compact. We shall now give descriptions of two eye-pieces in common use.

**Huyghens's Eye-piece.**—In this the two lenses are so arranged as to divide equally between them the deviation
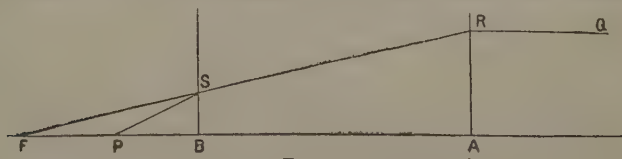


FIG. 104.

produced on a ray parallel and close to the axis. Let $f_1$, $f_2$ (negative quantities) be the focal lengths of the lenses at A and

B.  Let BA $= a$.  Suppose the ray Q R is deviated along R S F by the lens at A, and along S P by the lens at B. Then $\angle$ SPB $= 2\angle$ SFB.  Thus, since the angles are small, BP $= \frac{1}{2}$BF.  So, from the lens B we get, applying—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f},$$

$$-\frac{2}{BF} + \frac{1}{BF} = \frac{1}{f_2},$$

$$\therefore BF = -f_2.$$

$$\text{And } AF = -f_1.$$

$$\therefore a = -f_1 + f_2.$$

Thus the distance between the lenses is equal to the difference between the numerical values of their focal lengths. This is the condition that they should produce equal deviations in the ray Q R.

Further, the focal lengths of the lenses are taken to be in the ratio $3 : 1$, so that if $f$ is the focal length of B, $3f$ is the focal length of A ; and $a = -2f$.

Notice that $a = -\dfrac{f_1 + f_2}{2}$ in this case, so that the combination fulfils the condition of achromatism for excentrical pencils.  This was not designed by Huyghens ; it was pointed out by Boscovich.

This eye-piece is usually made of plano-convex lenses, the plane surfaces being next the eye.

Let us consider the way in which an image is formed by means of this eye-piece.  For the normal eye the rays which
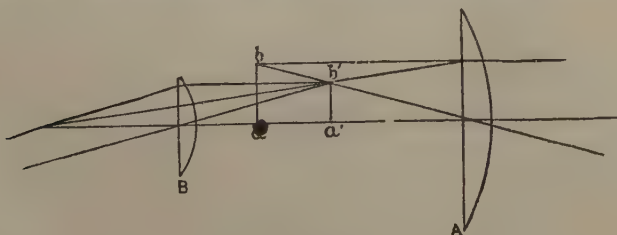


FIG. 105.

leave the eye-lens from any point of the image must be parallel, or the image at infinity, so that the eye may be quite at rest when the image is viewed.  Thus the image $a'$ $b'$, formed by A, must be at the principal focus of B, or midway between

the lenses. To find where the object is of which A produces this image, we have, since the distance of $a'$ from A $= -f$,—

$$\frac{1}{f} - \frac{1}{u} = \frac{1}{3f},$$

$$\therefore u = \frac{3f}{2}.$$

Thus the object with which this eye-piece deals is a virtual object, an image which would have been formed in a position behind that occupied by the field-lens. For this reason this eye-piece is sometimes called the **negative eye-piece.**

**Ramsden's Eye-piece.**—This is made of two lenses of equal focal lengths, the distance between them being two-thirds the numerical value of either focal length. The condition of achromatism is, then, not quite fulfilled, but the eye-piece is fairly achromatic. The lenses of which it is composed are usually two plano-convex lenses, with their convex surfaces facing each other.

Let us consider the formation of an image in this eye-piece. For a pencil to leave B as a parallel pencil, the image
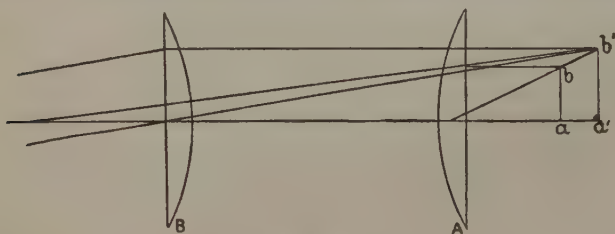


FIG. 106.

formed by A must be at $a'$, the principal focus of B. Let $f$ be the focal length of either lens. Then $Aa' = -\dfrac{f}{3}$; and to find the position of the object of which A forms the image $a'\,b'$, we have—

$$-\frac{3}{f} - \frac{1}{u} = \frac{1}{f}.$$

$$\therefore u = -\frac{f}{4}.$$

Thus the object with which the eye-piece deals may be a real object or a real image formed in front of the eye-piece by

some optical system. For this reason this eye-piece is some·times called the **positive eye-piece.**

In a microscope or telescope used for making measurements the eye-piece is fitted with fine wires or spider-lines. For this purpose the positive eye-piece is far more suitable. In this the wires are placed at $a'\, b'$, and are seen through both lenses, and so their image is corrected for chromatic and spherical aberration. In the negative eye-piece the wires at $a'\, b'$ would only be seen through one lens.

In the positive eye-piece the distance from the wires can be easily adjusted, to suit different eyes, by moving the eye-piece. In the negative eye-piece, when measuring-wires are affixed, they are usually fixed at the focus of the eye-lens, so as to suit the normal eye. A device is, however, occasionally employed by which the wires can be moved inside the eye-piece.

**Helmholtz's Formula for Ratio of Linear Dimensions of Object and Image.**—Helmholtz has given a formula showing the ratio of linear dimensions of an object and its image formed by refraction in a spherical surface, in terms of the angles of divergence from the axis of a ray before and after refraction.

Let A B, $a\, b$ be an object and its image formed by refraction in a surface of centre C. Let A R, $a$ R be paths of a ray before
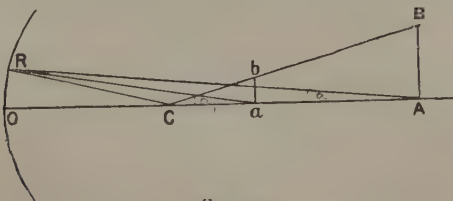
and after refraction; $\mu =$ index of refraction at surface. Then we have—

$$\frac{\text{CA}}{\text{OA}} = \mu\, \frac{\text{C}a}{\text{O}a}.$$

$$\text{Now } \frac{\text{CA}}{\text{C}a} = \frac{\text{AB}}{ab}.$$

$$\therefore \frac{\text{AB}}{\text{OA}} = \mu\, \frac{ab}{\text{O}a}.$$

Let the angles OAR, O$a$R be $a$ and $a'$.

Then, these angles being small—

$$a : a' = Oa : OA.$$
$$\therefore ABa = \mu aba'.$$

Or, if $\mu$, $\mu'$ are the indices of refraction of the first and second media, and $\beta$, $\beta'$ corresponding linear dimensions of object and image—

$$\mu\beta a = \mu'\beta'a'.$$

Now, we may suppose this image to produce another by refraction at another surface coaxal with the first, and this another at another such surface, and so on. Then the product $\mu\beta a$ is constant. So that if $\mu'$, $\beta'$, $a'$ refer to the last medium, we have, as above—

$$\mu\beta a = \mu'\beta'a'.$$

This formula is only correct for small values of $a$, $a'$; for the relation

$$a : a' = Oa : OA$$

only holds with this restriction.

We shall now make an application of this formula.

**Brightness of Image produced by Refraction at a Spherical Surface.**—Consider the object and image represented in the last figure; and suppose the image is seen by an eye suitably placed to see it, that is, along the axis.

Consider a small contour on the surface round O. Corresponding elements of object and image will send equal quantities of light through this contour. For it is the same light which reaches the surface in this contour from any part of the object that appears, after refraction, to come from the corresponding part of the image.

Now, the brightness of a surface which sends out a given quantity of light is inversely proportional both to the extent of the surface and to the solid angle through which it sends the light.

Corresponding surfaces of object and image are in the ratio $\beta^2 : \beta'^2$; and the solid angles subtended at these surfaces by the small contour in question round O are in the ratio $a^2 : a'^2$. Thus—

*Brightness of object : brightness of image*
$$= a'^2\beta'^2 : a^2\beta^2$$
$$= \mu^2 : \mu'^2 \text{ (by Helmholtz's formula).}$$

We may suppose successive images to be formed at

successive coaxal spherical surfaces. Then, *if no light is lost at the surfaces, or inside the media,* and $\mu$, $\mu'$ are the refractive indices of the first and last media—

$$\textit{Brightness of final image} : \textit{brightness of object}$$
$$= \mu^2 : \mu'^2.$$

It follows that the image seen through a lens (with air on both sides), or any number of lenses, is of the same brightness as the object.

This relation between the brightnesses I, I', of object and image, namely—

$$I : I' = \mu^2 : \mu'^2$$

has been proved on the supposition that the image is formed by means of small pencils along the axis joining object and image. The same relation, however, is true, no matter what the size of the pencils by which the image is formed. This is generally proved by saying that, if the relation were not true, we could cause an object to appear brighter by viewing it through a suitable combination of lenses—a result which is contrary to experience. We may, however, offer the following independent proof in the case of an image formed by refraction at a single surface :—

Suppose a small object, A B C, forms by refraction at a very small surface, R S, an image, A′ B′ C′. Let the projections at
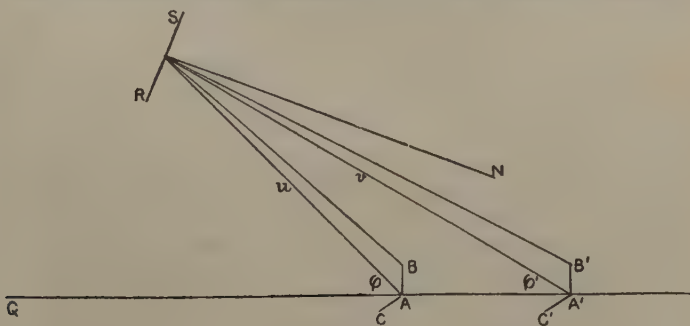


FIG. 108.

right angles to R A and R A′ of A B, A′ B′, the dimensions of object and image in the plane A A′ R, be $\beta$, $\beta'$; and let the projections of A C, A′ C′, the dimensions of object and image at right angles to this plane, be $a$, $a'$. Let R N be the normal at R to surface R S; R N is then in plane A A′ R. Let

$AR = u$; $A'R = v$; $\angle NRA = i$; $\angle NRA' = r$. Let the indices of refraction be $\mu$, $\mu'$. Then from formula—

$$\mu \sin i = \mu' \sin r,$$
$$\mu \cos i \, di = \mu' \cos r \, dr.$$

That is—

$$\mu \cos i \frac{\beta}{u} = \mu' \cos r \frac{\beta'}{v}.$$

Again, since C and C' are in a plane with R N, $a$, $a'$ subtend equal angles at the feet of the perpendiculars from C and C' on R N.

$$\therefore \frac{a}{u \sin i} = \frac{a'}{v \sin r},$$

or—

$$\frac{\mu a}{u} = \frac{\mu' a'}{v}.$$

$$\therefore \frac{\mu^2 a \beta \cos i}{u^2} = \frac{\mu'^2 a' \beta' \cos r}{v^2}.$$

Now, if I, I' are the brightnesses, or intrinsic luminosities, of A B C, A' B' C'; $s$, $s'$ their projected areas at right angles to $u$ and $v$; and S a small area of the surface at R; the quantities of light from A B C, A' B' C' passing across S are equal. Thus—

$$I s \cdot \frac{S \cos i}{u^2} = I' s' \frac{S \cos r}{v^2}.$$

$$\therefore I \frac{a \beta \cos i}{u^2} = I' \frac{a' \beta' \cos r}{v^2}.$$

$$\therefore \text{ from above, } I : I' = \mu^2 : \mu'^2.$$

Thus this relation holds between I and I', by whatever small pencils we suppose the image to be formed. If, then, A' B' C' is the image of A B C for any rays whatever, that is, if the refracting surface is an aplanatic surface for the object A B C, we always have the relation—

$$I : I' = \mu^2 : \mu'^2.$$

Suppose that the surface R S is part of a refracting surface of revolution about axis A' A Q. Then the small image A' B' C' is similar to the object. Thus—

$$a : a' = \beta : \beta'.$$

Then for the ratio in which the linear dimensions of the object are magnified, we have—

$$\frac{\mu a}{u} = \frac{\mu' a'}{v}.$$

And if the angles RAQ, RA'Q are $\phi$ and $\phi'$—

$$u : v = \sin \phi' : \sin \phi.$$
$$\therefore \mu a \sin \phi = \mu' a' \sin \phi'$$

We may extend these propositions, and show—

(1) That if an object forms a true image by refraction of rays at any number of surfaces, the separate surfaces not being aplanatic for the object, but the combination being so, and if $\mu$, $\mu'$ are the refractive indices of the first and last media, and I, I' the luminosities of the object and image—

$$I : I' = \mu^2 : \mu'^2.$$

(2) That if the aplanatic combination of surfaces is a series of surfaces of revolution about an axis on which the small object and its image are; $a$, $a'$ being the linear dimensions of object and image, and $\phi$, $\phi'$ the inclinations to the axis of corresponding rays in the first and last media, $\mu$, $\mu'$; then—

$$\mu a \sin \phi = \mu' a' \sin \phi'.$$

This last is a very important relation with regard to the action of optical instruments, and especially microscopes. It was given independently by Helmholtz and by Abbe.

**The Compound Microscope.**—In its simplest form this consists of two converging lenses with a common axis. One lens, turned towards the object to be viewed, is called the object-glass, or objective ; the other, at which the eye is placed to see the image, is called the eye-piece.

Fig. 109 shows the action of the instrument. The objective O forms of the object A B a real and much enlarged image, $a'$ $b'$. The eye-piece E forms of this image a virtual and enlarged image, $a b$. The magnifying power of the microscope is the magnification produced, $\frac{ab}{AB}$, when the image $a b$ is formed at the least distance of distinct vision. This distance differs from one eye to another ; but, that a definite magnifying power may be assigned to a microscope, the distance chosen is that for normal eyes, and is fixed at 10 inches.

Let this distance be D ; and let E$a$ be made equal to D. Let $m$ be the magnifying power. Then—

$$m = \frac{ab}{AB} = \frac{ab}{a'b'} \cdot \frac{a'b'}{AB} = \frac{Ea}{Ea'} \cdot \frac{Oa'}{OA}.$$

If $f'$ is the numerical value of the focal length of the eye-piece, so that the focal length is $-f'$, then—

$$\frac{\mathrm{E}a}{\mathrm{E}a'} = 1 + \frac{\mathrm{D}}{f'}.$$

And if $-f'$ is the focal length of the objective, the object must be situated just beyond the principal focus, so as to form an image, $a'\,b'$, very far from the lens as compared with its own distance, so that, very nearly—

$$\frac{\mathrm{O}a'}{\mathrm{OA}} = \frac{\mathrm{O}a'}{f}.$$

Other things being the same, the magnifying power will be inversely proportional to the focal length of the objective. Microscopes are generally supplied with various objectives, so that the magnifying power may be varied to suit the object under examination.

The magnifying power may be found, in practice, as follows: Let an object of known dimensions, such as a small scale, be viewed through the microscope. Place a scale at the distance D from the eye-lens aperture, and place a small reflector, such as a reflecting prism, just over this aperture, and so as to cover half of it, and so that, on putting the eye to the aperture, both the scales are seen—one through the microscope and the
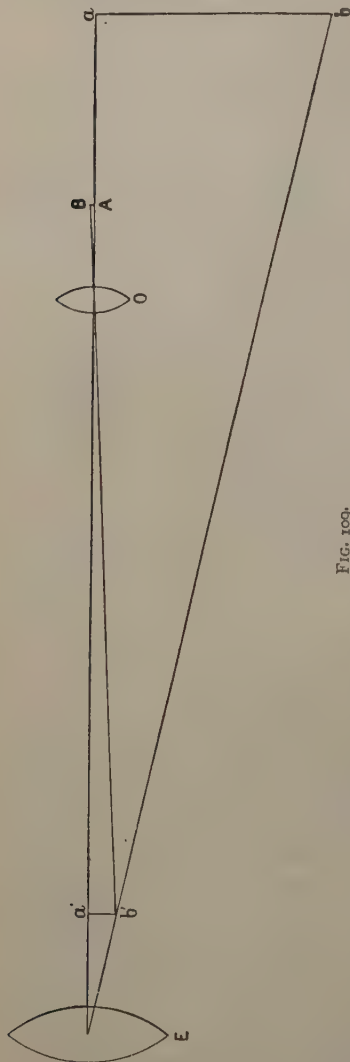
FIG. 109.

other by reflexion, the two images being superposed. Observe the lengths on the two scales that coincide in the images. The ratio gives the magnifying power. Thus, if $\frac{1}{100}$ inch in the small scale, seen through the microscope, coincides with an inch in the other, the magnifying power is 100. The two images should be truly coincident. But if the eye-piece is not quite set, so that the image formed by the microscope is at distance D, the result will not be materially affected; for, on account of the small focal length of the eye-piece, as the distance of the image varies, its magnitude practically varies in the same ratio, and it continues to subtend the same angle.

This is, however, very far from being a complete account of the modern microscope. We shall now consider a few points more in detail.

**Eye-piece.**—A microscope is generally fitted with one of the eye-pieces already described, according to the purpose for which it is to be used.

**Objective.**—Much attention has been given to the microscope objective; and a really good objective is an expensive article, and requires great care in making. An objective consists of a combination of lenses so arranged as to correct chromatic and spherical aberration. Many forms have been proposed and used for objectives. A form frequently used consists of three double lenses, each being an equi-convex lens of crown cemented to a plano-convex lens of flint, made achromatic for centrical pencils, the plane surfaces being all turned towards the incident light. In another form the front lens is a single plano-convex, nearly hemispherical, the plane surface being turned towards the light. This will, of course, show strong chromatic and spherical aberration. It is, therefore, combined with a strongly over-corrected system of lenses.

The correction for spherical aberration can be adjusted by relative motion of the lenses of the objective. Thus a part of the objective is made so that it can be moved with respect to the rest by means of an arrangement called the *screw-collar*.

The ordinary theories of spherical aberration and of formation of images by pencils not very oblique are totally inapplicable to the microscope objective, for it is necessary to have a great deal of light entering it, and we shall see that, on account of reasons connected with diffraction, it is necessary that the pencils should make all angles with the axis up to

very large values. Hence the angular aperture of modern objectives is made very large.

The cover-glass, or glass generally employed to cover a microscopic specimen, produces aberration of the more oblique rays from a given point O, as shown in the diagram. This produces an effect of the same sort as spherical aberration; and it can be corrected in the same way. The screw-collar must be adjusted to suit the particular thickness of cover-glass in use.
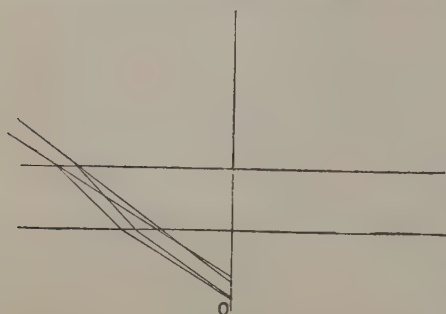


FIG. 110.

Fig. 111 shows how, of the rays coming from a point of the object and passing through the cover-glass and the air, some enter the objective and some are lost. If a liquid, having higher refractive index than air, is placed between the cover-glass and the objective, more of the rays will be gathered up; and even of those rays which before were received into the objective, through the air, less light will be lost by reflexion at the surfaces. There will thus be a gain in the amount of light received; and, what is of special importance with high-power objectives, there will be a gain of the more oblique rays. This arrangement is called *immersion*. The liquid which was at first used was water. But Abbe has discovered that oil of cedar-wood possesses about the same refractive index and dispersive power as the glass of the objective. It is, in consequence, the best fluid to use, and the arrangement is then called a **homogeneous immersion**.
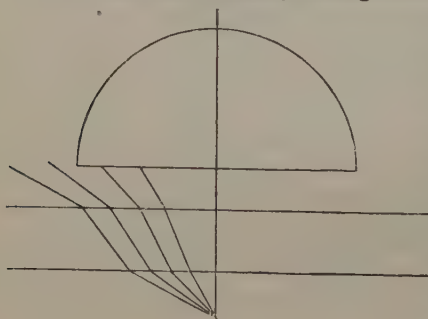


FIG. 111.

**Aperture of Microscope Objective.**—The angular aperture of the objective is the angle of the cone of rays sent by a point of the object into the objective. An essential point to aim at is that the aperture should be very large for the reasons already stated. Objectives are made with angular apertures in air as high as 110°. Hence the difficulty, with this great aperture, of making the objective at the same time free from spherical and chromatic aberration. There is, however, a better way of estimating the aperture of a microscope, not in terms of an angle at all. And this we shall shortly consider.

**Brightness of Image.**—We have seen that the image of an object and the object itself appear equally bright if both object and eye are in air; or if the object is in a medium of refractive index $\mu$, and the eye in air, the brightness of the image is to that of the object in the ratio $1 : \mu^2$.

To express more precisely the meaning of this statement, two small areas, S, S', of object and image, at corresponding points, P, P', and such that the orthogonal projections of S, S' at right angles to two straight lines, P T, P' T', are equal, will send equal quantities of light (when $\mu = 1$) down equal small solid angles about P T, P T'. Thus the quantities of light sent by S, S' into any small area, such as the pupil of the eye, placed in turn along and at right angles to P T and P' T', will be inversely proportional to the squares of the distances of S and S' from this area or pupil. Or, again, if the areas S, S' are taken to be such that their apparent sizes at the pupil are equal, they will send equal quantities of light into the pupil. This is expressed by saying that the object and image have equal brightnesses at the points P, P'.



Fig. 112.

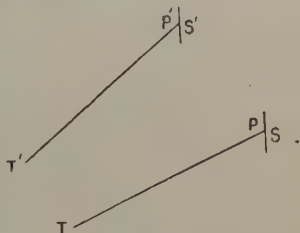Or, to express this in symbols, if I is the brightness of the area S, $\theta$ the angle which P T makes with the normal to S, and A the area of the pupil at distance $d$ from S and at right angles to P T, the light sent from S into the pupil is—

$$\frac{I \cdot S \cos \theta \cdot A}{d^2}.$$

Let the same letters, with dashes, be used to refer to S'; and the light sent from S' into the pupil is—

$$\frac{I' \cdot S' \cos \theta' \cdot A}{d'^2}$$

Now, $I = I'$; that is, if (1) $S \cos \theta = S' \cos \theta'$, the quantities of light received from the two areas are in the inverse ratio of the squares of their distances; and if (2) $\dfrac{S \cos \theta}{d^2} = \dfrac{S' \cos \theta'}{d'^2}$, or the *apparent sizes* are equal, that is, they subtend equal solid angles at the eye, the quantities of light received from them are equal.

Two surfaces, then, are equally bright if small areas, apparently equal when seen from two small equal apertures, held at right angles to the incident light, send equal quantities of light into these apertures. This, however, is on the supposition that the light sent from each surface fills the corresponding aperture; that is, the pencils of light must not be restricted to a part of the aperture only. Now, in the case of a high-power microscope, the pencils from the image are too narrow to fill the pupil, notwithstanding the great wideness of the pencils which leave the object, as the figure shows.

FIG. 113.

In this case, then, the brightness of the image as it appears to the eye, is to the brightness it would have if the pencils from it were wide enough to fill the pupil, in the ratio of the area of the pupil filled to the whole area of the pupil.

If a microscope reveals an object which was invisible without its assistance, it is never because it produces an image brighter than the object. The image is, at best, only as bright as the object; and in practice is less bright. But it is because of its increased apparent size, or
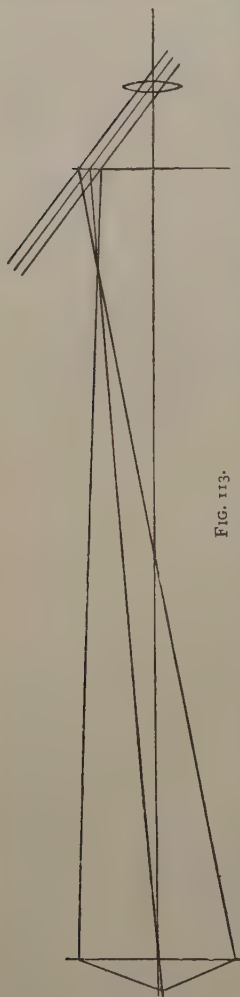
solid angle subtended at the eye; though the brightness is diminished, the light received is increased, and is enough to produce an appreciable effect on the eye, whereas that from the object was not. The microscope accomplishes this by gathering up rays from the object which would not have entered the pupil, and directing them towards the pupil, as will be understood on examining the figure.

Imagine two corresponding elements of object and image on the axis. Let $a, a'$ be the divergences from the axis of the extreme rays through these. Let $\beta, \beta'$ be corresponding linear dimensions of object and image. Let $\mu$ be the refractive index of the medium in which the object is. Suppose the image is at a distance, D, from the eye, the distance of distinct vision. Let $p$ be the radius of the pupil. The light entering the eye by a pencil from the object is to that which would enter if the pupil were filled, in the ratio—

$$\pi D^2 \sin^2 a' : \pi p^2.$$

Thus if I is the brightness of the image, and I' that which the image would have if the pupil were filled—

$$I = I' \frac{D^2}{p^2} \sin^2 a'.$$

$$\text{But } \mu\beta \sin a = \mu'\beta' \sin a' = \beta' \sin a'.$$

$$\therefore I = I' \frac{D^2}{p^2} \mu^2 \sin^2 a \cdot \frac{\beta^2}{\beta'^2}$$

$$= I' \frac{D^2}{p^2} \cdot \frac{\mu^2 \sin^2 a}{m^2},$$

where $m$ is the magnifying power of the instrument.

Suppose the object to be in air, and separated from the front of the objective by a cover-glass and a liquid for immersion, the surfaces of the cover-glass being planes perpendicular to the axis, the quantity $\mu \sin a$ will be the same for a ray proceeding from the object in the air, the glass, and the liquid. Thus the quantity $\mu \sin a$, where $\mu$ is the refractive index of the medium just outside the objective, air or liquid, and $a$ is the greatest angle made with the axis by rays in this medium, determines (other things being equal) the brightness of the image, and it is essential to make this quantity great. This quantity is called the **numerical aperture** (N.A.) of the instrument.

The same quantity may be expressed in terms of the focal length of the objective, and the diameter of the cone of light

which passes through it. Let $f$ be the numerical value of the focal length; and $b$ the radius of the circle of section of the cone of light by the second focal plane of the objective. This is practically the same as the semi-breadth of the cone of light taken at the back surface of the objective; for this surface very nearly coincides with the second focal plane in all ordinary objectives. Now, let $u$ be the distance of the image formed by the objective from the second principal focus, $\beta'$ the size of this image corresponding to $\beta$ in the object, $\alpha'$ and $\alpha$ the divergence of extreme rays in image and object from the axis. Then—

$$\frac{\beta'}{\beta} = \frac{u}{f}.$$

$$\mu\beta \sin \alpha = \mu'\beta' \sin \alpha' ;$$

$$\therefore \mu \sin \alpha = \mu'\frac{u}{f} \sin \alpha'.$$

But $\mu' = 1$, and $u \sin \alpha' = b$, practically.

$$\therefore \mu \sin \alpha = \frac{b}{f}.$$

A microscopic specimen is frequently kept immersed, under its cover-glass, in some liquid, such as balsam or glycerine. This prevents loss of light by reflexion, and to some extent corrects inequalities in the under surface of the glass. Since brightness of image $= \frac{1}{\mu^2}$ of brightness of object, this arrangement might appear to be prejudicial to the brightness of the
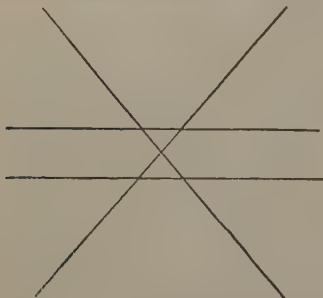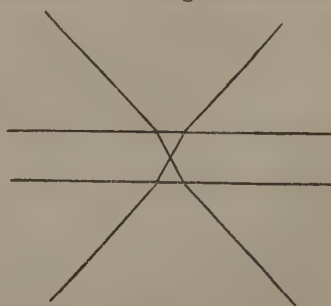


FIG. 114.         FIG. 115.

image. But the brightness of the object, or the light that comes from it, will depend on the light it receives. Suppose the object to be opaque, and we wish to view it against a

bright field formed by light from the under side, as is frequently
the case; Figs. 114 and 115
show, respectively, how a cone
of light would illuminate a dry
object and an object immersed
in liquid, and pass away from it
on the upper side, as much light
ultimately diverging from the
object above the cover-glass, in
a cone of given angle, in each
case. In fact, the real bright-
ness of the *object* is greater in the
second case, because it receives
more light, and hence sends out
more, per unit of solid angle.

Let O C O', E D E' denote
object-glass and eye-piece. Let
$a\,h\,b$ be the image formed by
O O' of an object A H B. Let
H be the point in the line A H B
furthest from A, the *whole* pencil
of light from which, after refrac-
tion through O O', falls on E E'.
The figure shows how a portion
only of the pencil from B falls
on E E', and as points are taken
along A H B further off from A,
less light from these points will
be received by the eye, or the
images of these points become
more and more dim; that is, the
edge of the image gradually
fades out, and there is produced
what is called the *ragged edge*.
To prevent this, a diaphragm is
put at $a\,h\,b$, with a circular aper-
ture of radius $a\,h$, and this is
called the *stop*.

With a positive eye-piece the
position for the stop is in front
of both lenses, at the focus of
the eye-pece; with a negative
eye-piece it must be placed
between the lenses at the focus of the second, or eye-lens.



FIG. 116.

Let $o\,c\,o'$ be the image, formed by the eye-piece, of O C O. Then any ray can be easily drawn after passing through E E,' for it passes through the point of $o\,c\,o'$ corresponding to the point of O C O' from which it came. The rays from H which meet the object-glass at the points O, C, O' are drawn. They meet $o\,c\,o'$ in the points $o, c, o'$. The area $o\,c\,o'$ is the region of greatest conc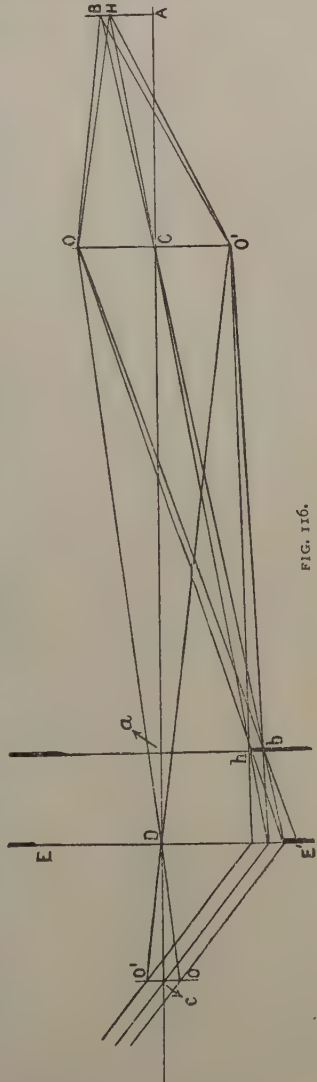entration of rays. It is called the **eye-ring.** The point $c$ is called the **eye-point.** If the pupil of the eye is of diameter $o\,o'$ or less, it receives most light when at the eye-ring. On moving the eye from this position in either direction, rays will be lost, chiefly from points at the outer edge of the field. If the pupil is of greater diameter than $o\,o'$, it may be moved a little from $c$ without loss of light ; but at $c$ all the pencils enter it centrically. The eye-ring is thus the best position for the pupil.

**Micrometer Eye-piece.**—This is used to measure the dimensions of an object. It consists of a positive eye-piece with a fine wire in its focal plane, which wire is capable of being moved transversely in this plane by means of a screw carrying the frame to which it is attached. The screw-head is divided so that we may estimate the distance in terms of the screw-pitch through which the wire has been moved across the image formed by the object-glass. Now, this image is similar to the object, so that the distances through which the wire is moved are proportional to the corresponding distances in the object. To measure a distance in the object, then, we must estimate the corresponding distance in the image in terms of the screw-pitch, and then replace the object by a small scale, such as a subdivided millimeter, and estimate a length on the image of this in terms of the screw-pitch. Thus we get, by proportion, the required distance.



FIG. 117.

This arrangement would not be applicable to the negative eye-piece, because the image formed at the position in which the wire must be put is not similar to the object, on account of the linear distortion produced by the field-lens.

**Binocular Microscope.** — This has two eye-pieces for the two eyes, with axes converging to the object-glass. The light leaving the object-

glass is divided so as to form images in both eye-pieces. There are various ways of dividing the light. Fig. 117 illustrates how it may be done; one half of the pencil coming from the object-glass being received by a suitably shaped piece of glass with plane faces, and, after two internal reflexions, emerging in a direction inclined to that of the other half. The distance between the eye-pieces has to be arranged to suit the distance between the eyes. This may be done by having the eye-pieces movable in their diverging tubes by means of rackworks.
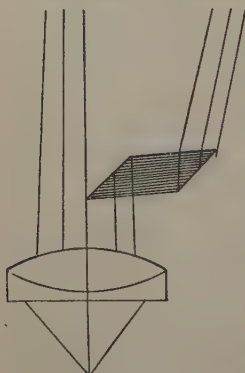
**The Astronomical Telescope.**—In its simplest form this consists of two converging lenses—an object-glass and an eye-piece. The object-glass, O, forms an image, $a'\,b'$, of a distant object, A B. $a'\,b'$ is nearly at, just beyond, the principal focus. The eye-piece E forms an enlarged virtual image, $a\,b$, of $a'\,b'$.

Let $-f, -f'$ be the focal lengths of object-glass and eye-piece. The *magnifying power*, $m$, is the ratio of the angles subtended at the eye by $a\,b$ and by A B, when they are both at great distances from the telescope. The exact position of $a\,b$ does not practically affect this ratio; for $f'$ is so small that $a'\,b'$ will be practically at the principal focus of E. The exact position of A B does not matter practically; for as long as A B is very far off, the image $a'\,b'$ will be practically at the principal focus of the object-glass. The angle which A B subtends at the eye is practically equal to A O B, *i.e.* to $a'$ O $b'$. Putting tangents for angles, we have—



Fig. 118.

$$m = \frac{a'b'}{Ea'} \div \frac{a'b'}{Oa'} = \frac{Oa'}{Ea'} = \frac{f}{f'} \text{ practically.}$$

The magnifying power of a telescope may be found by looking through it at a distant scale or some object with equidistant marks, such as a brick wall, with one eye, while the object is seen directly with the other eye. The two pictures thus seen are got to overlap, and the number of divisions in the latter that coincide with one division in the former is the required magnifying power.
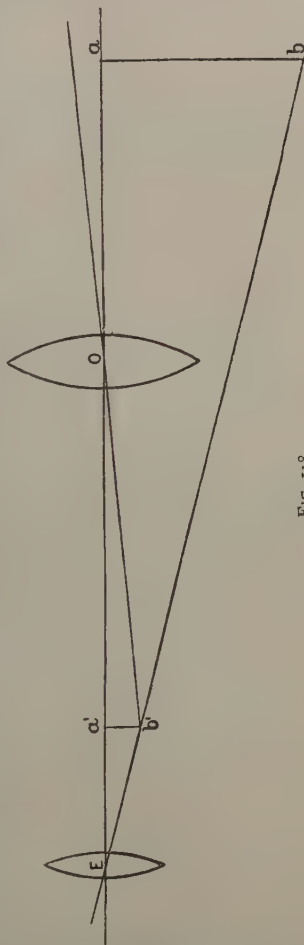
Another method of finding the magnifying power is as follows: $m = \frac{f}{f'} =$ (by expression for magnification produced by lens) diameter of object-glass $\div$ diameter of image formed by eye-piece of object-glass. Thus it is only necessary to measure the object-glass, or some object in the place of it, such as the length of a slit cut in a piece of cardboard; and the length of the corresponding image.

In an actual telescope the parts are not so simple as here described. The object-glass is generally a compound lens, made achromatic for centrical pencils, consisting of a convex lens of crown and a concave of flint glass. The eye-piece of an astronomical telescope, or of an ordinary telescope used for observing purposes, is generally one of those already described. This arrangement, however, produces an inverted image. To avoid this an erecting eye-piece is used. One form of this consists of four convex lenses, the first two of which are of equal focal length, and so placed that the image formed by the objective is at the first principal focus of the first lens. Then an image would be formed at the second principal focus of the second lens. The other two lenses form an ordinary Huyghens eye-piece.

As with a microscope, a stop is used to cut off the ragged edge; and if the telescope is to be used for observations of position, or measurements, it is furnished with cross-wires in the focus of the eye-piece, or eye-lens.

In telescopes, as in microscopes, if the magnifying power is high, the pencils of light do not fill the pupil. If the eye-ring were as large as, or larger than, the pupil, the brightness of the image to an eye with pupil at eye-ring would be the same as that of the object; $= I'$, say. Let I be the brightness of the image as it appears to the eye. Let $r, p$ be the radii of eye-ring and pupil, $r$ being less than $p$. Then—

$$I = I' \left( \frac{r}{p} \right)^2.$$

This can be put into a different form. For if $m$ is the magnifying power of the instrument, and $b$ the radius of the object-glass, $m = \dfrac{b}{r}$; or $r = \dfrac{b}{m}$. Thus—

$$I = I'\left(\frac{b}{mp}\right)^2.$$

We see, then, the advantage of having a wide object-glass. The entire quantity of light received from an object through the telescope will be to that received by the eye without the telescope in the ratio $Im^2 : I'$; for the light received per unit of solid angle is changed in the ratio $I : I'$, and the solid angle is changed in the ratio $m^2 : 1$. Thus the light is increased in the ratio $b^2 : p^2$, that is, in the ratio of the aperture of the object-glass to that of the pupil.

This, too, is otherwise obvious; for without the telescope the eye would only have received as much light as could enter the pupil, but the telescope gathers up all that enters the object-glass and directs it into the pupil. Thus stars which are invisible to the naked eye are revealed by a wide-aperture telescope.

The *field* of view of a telescope means the angle subtended at the eye by the diameter of the whole area of the object seen; or the angle between the extreme pencils from the object to the eye. This is practically the same as the angle subtended at the centre of the object-glass. If it were determined by pencils whose axes meet the eye-piece, it would be twice the angle $b\,C\,a$ in Fig. 116; that is, the angle subtended at the centre of the object-glass by the diameter of the eye-piece. It is, however, limited by the stop to those pencils all of whose rays meet the eye-piece. Thus it is twice the angle $h\,C\,a$. Let $b, b'$ be the radii of object-glass and eye-piece (or field-lens of eye-piece). Let $x =$ radius of stop, and $2\Theta =$ field. Then, by drawing a straight line from O parallel to C D to meet E E', we have, by similar triangles—

$$\frac{x + b}{f} = \frac{b' + b}{f + f'};$$

$$\therefore x = \frac{b'f - bf'}{f + f'}.$$

And $\Theta$ being small, $= \tan \Theta$.

$$\therefore \Theta = \frac{b'f - bf'}{f(f + f')}.$$

**Galileo's Telescope.**—This consists of a convex lens for object-glass, and a concave lens for eye-piece. The image which the object-glass would form is behind the eye-piece; so that it is never formed, but acts as a virtual object to the eye-piece. The eye-piece reinverts this in the final image, thus giving an erect image; and this is the main advantage of the instrument. The figure shows the instrument arranged to produce an image $a$, at an infinite distance, of an object A at an infinite distance; $a'$ being the image of A formed by the object-glass. $a'$ is in a focal plane of both O and E. Thus the distance between O and E is the difference of their numerical focal lengths. Or, as for the astronomical telescope, $OE = -f_e - f_o$. In this telescope no stop or cross-wires for measuring can be used. The eye-ring is in front of the eye-piece, so that the eye cannot be put at it; the rays diverge from the eye-piece, and the field is very limited. The area of the eye-piece used, or the effective aperture, is equal to the area of the pupil.
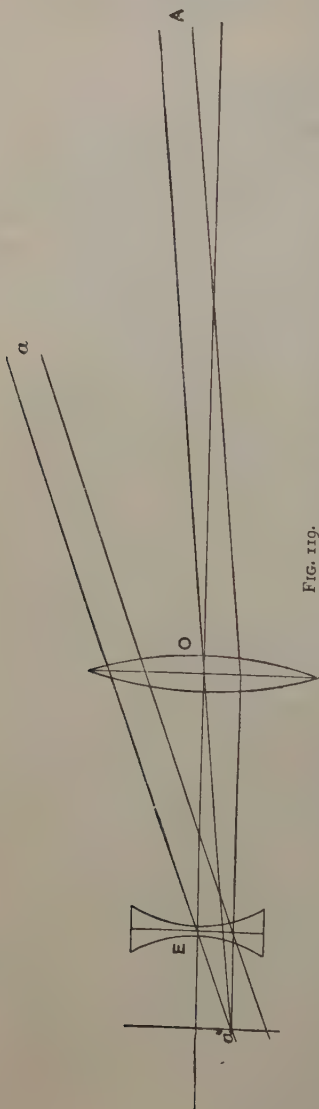
The opera-glass consists of a pair of Galilean telescopes, the object-glass and eye-piece of each being, as a rule, compound achromatic lenses, with an arrangement for varying the distance between object-glass and eye-piece, for different eyes and different distances of object.

FIG. 119.

**Reflecting Telescopes.**—In all these, instead of an object-glass consisting of a convex lens, a concave mirror, called the object-mirror, is used to form an image of the object.

In *Herschel's* telescope the aperture is very large, and the
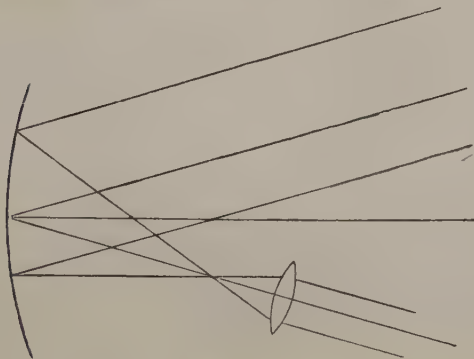


FIG. 120.

image formed by the mirror is viewed directly with an eye-piece. The incidence is made slightly oblique, and the eye-piece placed a little to the side, so that the head of the observer may be as much out of the way as possible.

In *Newton's* telescope the incidence on the concave mirror is direct, and a small plane mirror or reflecting prism is used to reflect the image out at the side, where the eye-piece receives it.
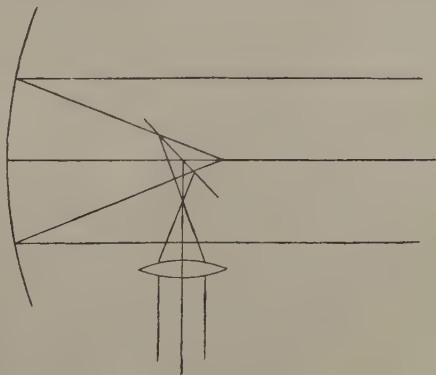
In *Gregory's* telescope (Fig. 122) the light, after reflexion at the mirror, forms an image, $a' b'$. This



FIG. 121.

image forms, with the help of another concave mirror, an image, $a'' b''$. This is viewed by the eye-piece. Adjustment for different eyes is effected by moving the small mirror along the axis.
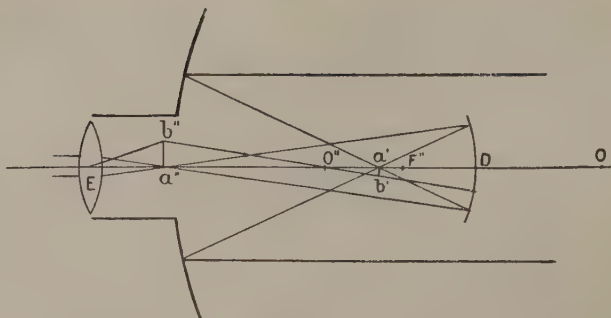
FIG. 122.

In *Cassegrain's* telescope (Fig. 123) there is a small convex mirror instead of the small concave mirror of Gregory's. The image $a'\,b'$ is a virtual object for this mirror, and gives the real image $a''\,b''$. The image as seen by the eye-piece is then inverted.
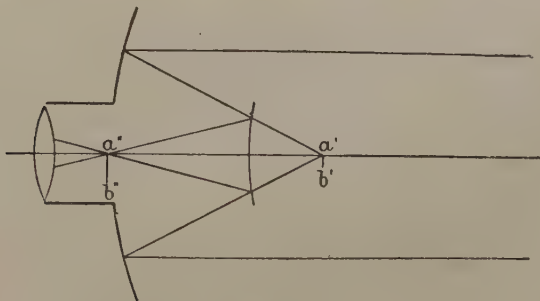


FIG. 123.

To find the magnifying power of Herschel's telescope. Let the distant object A B form in the object-mirror the image $a'\,b'$. Then—

$$m = \frac{\angle\, a'\mathrm{E}b'}{\angle\, \mathrm{ACB}}$$
$$= \frac{\angle\, a'\mathrm{E}b'}{\angle\, a'\mathrm{C}b'} = \frac{a'\mathrm{C}}{a'\mathrm{E}} = \frac{f}{f'},$$

where $f$ and $-f'$ are the focal lengths of object-mirror and eye-piece.

In Newton's telescope let $a'\beta'$ be the image formed by the object-mirror, $a'\,b'$ the image formed by the plane mirror. Then $a'\beta' = a'b'$, and we have—

$$m = \frac{\angle \, a'Eb'}{\angle \, ACB}$$

$$= \frac{\angle \, a'Eb'}{\angle \, a'C\beta'} = \frac{a'C}{a'E} = \frac{f}{f'}.$$

Let us consider the magnifying power and the field of view of Gregory's telescope. Let $f$, $-f'$, $f''$ be the focal lengths of the large mirror, eye-piece, and small mirror. Let O, O'' be the centres of the large and small mirrors, F'' the focus of the small mirror. Let the distance between the foci of the mirrors be $x$.

The magnifying power—

$$m = \frac{\angle \, a''Eb''}{\angle \, a'Ob'} = \frac{a''b''}{a'b'} \cdot \frac{Oa'}{Ea''} = \frac{a''O''}{O''a'} \cdot \frac{Oa'}{Ea''}.$$

Then $Ea'' = f'$; $O''a' = f'' - x$; $Oa' = f$. And—

$$\frac{1}{Da''} + \frac{1}{Da'} = \frac{1}{f''};$$

$$\therefore \frac{1}{2f'' + O''a''} + \frac{1}{f'' + x} = \frac{1}{f''}.$$

$$\therefore O''a'' = \frac{f''(f'' + x)}{x} - 2f''$$

$$= \frac{f''(f'' - x)}{x}$$

$$\therefore m = \frac{f''f}{xf'}.$$

We may find, from this, an approximate expression not involving $x$.

The image $a''b''$ is near to, and we shall suppose it at, the pole of the large mirror. Thus—

$$f = a''a' = a''O'' + O''a' = \frac{f''(f'' - x)}{x} + f'' - x;$$

$$\therefore fx = f''^2 - x^2.$$

$x$ being small, we have, approximately—

$$fx = f''^2;$$

$$\therefore m = \frac{f^2}{f'f''}.$$

Supposing the object-mirror large enough, so that it imposes no restriction on the field, the field will be limited by the extreme rays drawn from the small mirror to the eye-piece. The extreme visible field is found by drawing a straight line between

the highest points of small mirror and eye-piece. Now, the small mirror is made about equal in size to the aperture in the object-mirror, so as just to cut off all incident rays that would fall directly on the eye-piece, and the eye-piece is made of about the same size to take in as great a field as possible. Thus if $a$ is the radius of the front lens of the eye-piece, and $a'' b''$ is the radius of the image, formed by the small mirror, of the whole object seen, $a''b'' = a$, about. And the semiangle subtended by the image seen is $\dfrac{a}{f}$. Therefore the field is—

$$\frac{2a}{fm},$$

$m$ being the magnifying power of the instrument.

The two expressions here found for the magnifying power, and the expression for the field of view, also apply to Cassegrain's telescope.

In the reflecting telescope the difficulty of obtaining an achromatic object-glass is got over once for all. Such a telescope does not produce so bright an image or such good definition as an equally large refracting telescope; but mirrors for reflecting telescopes can be made much larger than object-glasses. Lord Rosse's large reflecting telescope, which can be used either as a Newtonian or as a Herschelian, has a reflector of 6 feet diameter, and 53 feet focal length; whereas object-glasses have not been made of 2 feet diameter. The material frequently used for object-mirrors is speculum metal, a compound of copper and tin. Mirrors made of this material, however, are found difficult to keep in good order, as it readily becomes tarnished, and great care has to be exercised in cleaning and polishing it, so as not to alter the shape of the mirror. Reflectors have also been made of silver deposited on glass, the silvered side being used as the reflecting side. These afford a better reflecting surface, are more easily made, and more easily kept in order.

Among large telescopes that have been constructed on the reflecting principle, in modern times, may be mentioned that of the Melbourne Observatory, which is a Cassegrain telescope of 4 feet aperture. This form is not frequently used. As a terrestrial telescope, it possesses the defect of giving an inverted image. But it has the advantage over Gregory's, which produces an erect image, that the spherical aberrations of its mirrors are in opposite directions, and so tend to cancel each other.

**The Spectrometer.**—This is an instrument used for measuring the refracting angles and refractive indices of prisms. Its main parts are a horizontal divided circle; a horizontal platform turning about the axis of the divided circle, and having a vernier, or two verniers at opposite ends of a diameter, which move over the circle; a collimator set horizontally, at a higher level than the platform, and turned towards the axis of the circle; a telescope set horizontally at the same level as the collimator, turned towards the axis of the circle, and movable about this axis, with verniers moving over the circle. The collimator consists of a convex lens at the end of a tube, with a vertical slit, or a pair of cross-wires, in the focus; the telescope is a common telescope with a positive eye-piece, and a pair of cross-wires in the focus of the eye-piece. In both collimator and telescope the distance of the cross-wires, or of the slit in the collimator, from the object-glass is adjustable. And in the telescope the distance of the eye-piece from the cross-wires can be adjusted to suit the eye.

To use the instrument, first set the telescope eye-piece to suit the eye. Then the telescope and collimator must be set for parallel light. This may be done by setting the telescope first, turning it to a very distant object, and adjusting it till the image of this coincides with its wires. Then turn the telescope to the collimator; and adjust this so that the image of its slit or wires coincides with the wires of the telescope. Suppose that cross-wires are used in the collimator; then the cross-wires in telescope and collimator are at the principal foci of the object-glasses. The pencils of light from the collimator wires leave the collimator as parallel pencils, and on entering the telescope converge to the plane of its wires.

To measure the angle of a prism, this is set on the platform with its faces vertical, or at right angles to the plane in which the axes of the telescope and collimator are. The best way of doing this is to have the prism on a little table provided with three levelling-screws, and resting on the platform, and so to adjust the level that the image of the collimator wires occupies the same position in the field of the telescope when seen by reflexion at either face as when seen directly. In making this adjustment we may either fix the telescope on one side, not in a line with the collimator, and turn the platform with the prism so as to reflect light from the collimator into the telescope off the faces in turn; or we may fix the prism in such a position that each face reflects some of the light from the collimator, and turn the telescope so as to receive light off each face in

turn. These adjustments being made, we may measure the angle of the prism in two ways.

1. Fix the prism so that each face reflects light from the collimator. Turn the telescope to receive light from one face of the prism and set it so that the image of the collimator
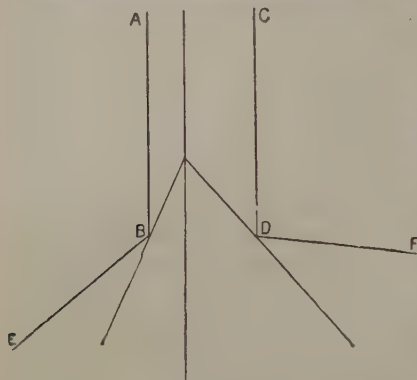


FIG. 124.

wires is formed in a definite position with reference to its wires, so that, for instance, the intersections of the two pairs coincide. Read the angle at which the telescope is set. Move the telescope till the image of the collimator wires is seen by reflexion off the other face of the prism in the same way. The angle of the prism is half the angle through which the telescope has been turned. For if A B, C D denote the parallel rays of light coming from the collimator, and B E, D F the reflected rays; these last will be the directions along which the telescope axis is set in the two positions; and they contain an angle equal to twice the angle of the prism.

2. Set the telescope on one side, and turn the prism so as to reflect the images of the collimator wires in turn into the telescope. The angle through which the prism is turned between the two positions is the supplement of its refracting angle.

To measure the refracting index of the prism, we have, in addition, to determine the angle of minimum deviation which the prism produces. This must, of course, be done with light of the definite quality for which we require the prism's refractive index—say, for sodium light got by burning a sodium salt in a Bunsen flame. The most accurate way of determining the angle is as follows: Set the prism to produce minimum deviation on one side, and set the telescope to view the image, taking the reading of the position of the telescope. Now turn the prism till it produces minimum deviation on the other side, and set the telescope to view the image. The angle of minimum deviation is half the angle through which the telescope has been turned between the two positions.

**The Sextant, or Quadrant,** invented by Newton, and reinvented by Hadley, is used to measure the angular distance between two distant objects. It depends on the principle that if a beam of light is reflected by a mirror, then, when the mirror is rotated, the reflected beam is rotated through twice as great an angle ; or if a given beam is to be reflected in a given direction, then, when the incident beam is rotated, the mirror must be rotated through half as great an angle.

M and N are two mirrors set at right angles to the plane of the instrument ; only half of N is silvered, the remainder being transparent. M is movable about an axis at



FIG. 125.

right angles to the plane of the instrument by means of the arm M P, which moves over a divided arc of a circle, and carries a vernier at P. T is a telescope turned to the mirror N. If M is parallel to N, rays of light would enter T from the same distant point, both through the transparent part of N and after reflexions in the two mirrors. P should now be at the zero of the divided arc. To measure the angular distance between two distant objects, M is turned till the two objects are seen to coincide in the telescope. The angle through which M is turned, given by the reading at P, is half the required angular distance. The instrument is generally graduated so as to give the angular distance directly. Thus the arc is numbered in half-degrees.

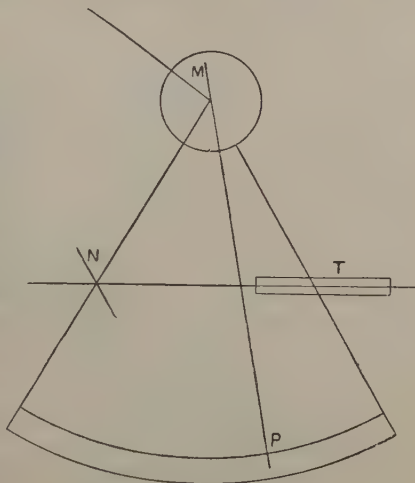**The Heliostat.**—It is frequently necessary to have sunlight in a definite direction for a considerable time. The heliostat is an instrument which, by means of clockwork, keeps a reflector turning so as to accomplish this. The position for the mirror, to reflect the light of the sun in a given direction, is such that its normal bisects the angle between the direction and the direction of the sun. There

must then be some arrangement for causing the mirror to move so that its normal always has such a position. It will be sufficiently accurate to suppose that, during the time for which a heliostat is used, the sun's polar distance—that is, the angle between the sun's direction and the earth's axis—remains constant, and that the sun moves round this axis, relatively to the earth, with uniform angular velocity. The original and simplest form of heliostat is that of Fahrenheit. In it is an axis which is set parallel to the earth's axis, this being done by setting it in the meridian, and to make an angle with the horizon equal to the latitude of the place. The mirror can be firmly attached to this axis, and is set so that its normal makes an angle with it equal to half the sun's polar distance. The axis is carried round by clockwork at the rate of one revolution per day, in the sense of the sun's motion. If, then, the mirror is set so that its normal is, on starting, in the plane containing the sun and the earth's axis, it will be in such a plane always; and the mirror will turn so as to continually reflect the sun's light along the axis. Another reflector could then be fixed, if necessary, so as to send the sun's rays in any required direction.

**Silbermann's Heliostat.**—As an example of a heliostat, we here describe that of Silbermann. The drawing and the description are taken from Jamin's "Cours de Physique." $m\,m$ is the mirror which reflects the sun's rays in the required direction. Let O S be the direction of the sun, and O R that in which it is required to reflect the light. The apparatus is so arranged that the normal O N to the mirror shall always bisect the angle R O S, and then the sun's light is always reflected along O R. $m\,m$ is carried on a jointed framework, in which $p\,e\,f\,a$ forms a rhombus, $p\,f$ being its diagonal; $p\,f$ is rigidly attached to the mirror, and is normal to its surface. $p\,e$ is parallel to the rod $r$, which is set in the direction O R. $p\,a$ is parallel to the rod. Arrangements have thus to be made to cause the rod C always to point towards the sun. For this purpose this rod is carried by the arm C P $s$, which consists of a portion of a divided circle having its centre at O, and is rotated, by means of the clockwork in the box H, about the axis F at the rate of one revolution in 24 hours. The box is set so that the axis F is inclined to the horizon at an angle equal to the latitude of the place in which the apparatus is used, and so that this axis is in the meridian; and is kept in this position by the clamping-screw K. The limb C D $s$, which slides in an aperture at D, is then set so that the angle

C O D is equal to the complement of the sun's declination, that is, to the angle which the direction of the sun makes with the earth's axis. If, then, the clockwork being set going, C O is set to point towards the sun, it will continue to do so.
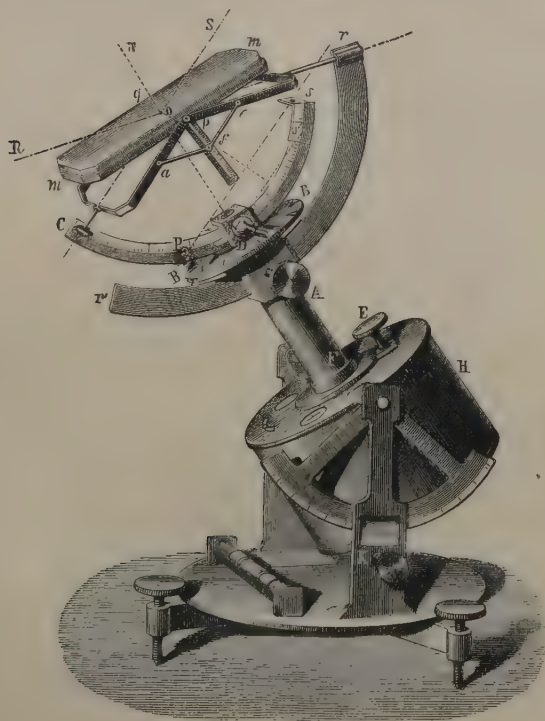


FIG. 126.

C O may be turned towards the sun by turning C s till a needle, which is attached to the axis at D, indicates the hour of *true* time on the dial B B. The limb r r' is carried by a hollow support which surrounds the axis that connects C P s with the clockwork, and moves independently of this axis; by means of the two clamping-screws E and A, the rod r can be set in any desired direction, the direction in which the light is to be reflected.

Attached to the limb C P s are a pin-hole aperture s and a small screen P, such that the line P s is parallel to C O.

M

These may be used in setting the apparatus. It is, for instance, unnecessary to know the meridian plane if the apparatus is set correctly in all other respects. For if this is the case, and the whole apparatus be turned about, keeping its base level, till the sun shines through *s* on P, then the axis must be in the meridian.

## CHAPTER IX.

### *VELOCITY OF LIGHT.*

THE first successful attempt to measure the velocity of light was made by Roemer, a Danish astronomer, in about 1675. His method was based on observations of the eclipses of Jupiter's first satellite. The plane in which the satellite revolves round Jupiter is nearly the same as that of Jupiter's orbit about the sun; and the satellite becomes periodically eclipsed, passing into the shadow of Jupiter cast by the sun. The satellite revolves uniformly about Jupiter, as does Jupiter about the sun, so that successive eclipses should occur at equal intervals, and so should successive emergences. In the figure S denotes the sun, and the inner and outer circles the orbits of the earth and Jupiter. Suppose both to move in the plane of the paper, clockwise. Jupiter's period of revolution is 11 years and 10 months. Suppose the planets (E and J) to be in conjunction, that is, in the positions $E_1$, $J_1$. In a little more than 6 months they will be in opposition, at $E_2$, $J_2$. And after another equal interval they will be in conjunction again at $E_3$, $J_3$. Now, while E moves from $E_1$ to $E_2$ the appearances of the satellite can be observed; and while E moves from $E_2$ to $E_3$ its disappearances can be observed. It is found that the mean interval between successive appearances is longer than the mean interval between successive disappearances. This is because light takes a finite time to reach the earth from the satellite, and a longer time as the earth gets further away. If the interval between successive disappearances or appearances is calculated from observations made as the earth moves from conjunction to conjunction again, it is found

that the interval between appearances at $E_1$ and $E_2$ is 16 mins. 26 secs. longer than the calculated time, and the interval between disappearances at $E_2$ and $E_3$ is by the same amount shorter than the calculated time. Thus 16 mins. 26 secs. is
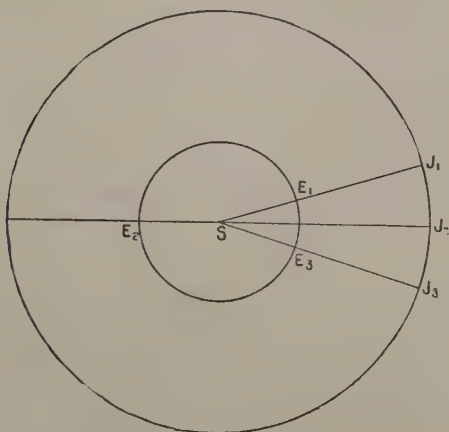


FIG. 127.

the time that light takes to travel over the diameter of the earth's orbit. And the velocity of light in space is found by dividing this diameter by 16 mins. 26 secs. The result that has been obtained in this way is 308,000,000 metres per second.

Bradley, the English astronomer, in 1728, explained, by means of the finite velocity of light, the astronomical pheno- menon called *aberration;* and deduced from the phenomenon the value of the velocity of light in space. This pheno- menon is a small apparent periodic dis- placement of the fixed stars from the mean position. The period of the displacement is a year; and the displacement is always in the direction in which the earth is moving in its orbit. Now, suppose the earth to be at A, and let A B denote the velocity with which it is moving; let light be coming to A along the path C A and with a velocity denoted by C A; then



FIG. 128.

the relative approach of the light towards the earth is com- pounded of the two velocities C A and B A, that is, it is

represented by D A. That the light will appear to come along the path D A will, perhaps, be made clearer by the following illustration :—

Suppose a telescope (and the same reasoning would apply to the eye) to be directed along A C. Then, if the light enters one end of the tube at C, by the time the light has arrived at A, the other end of the tube will have got to B, and the light will not be perceived. But if the telescope is directed along A D, the light coming along D B enters the tube at D, and by the time it arrives at B, the other end of the tube is there to receive it. Thus the light appears to come along D A. This explanation does not at all depend on the nature of light, but merely on the facts that light travels with finite velocity, and along straight lines, so that its path in the moving telescope tube is in a straight line with all the remainder of its path.

Let, then, α denote the angle which the apparent distance of the star is making with the earth's path, that is, the angle C A B ; and let δ denote the angular apparent displacement of the star from its true position. Let V and $v$ be the velocities of light and of the earth. Then we have—

$$\frac{v}{V} = \frac{\sin \text{BDA}}{\sin \text{BAD}} = \frac{\sin \delta}{\sin \alpha}.$$

To find δ it is necessary to know the true direction of the star. This, for many stars, is the same as its mean apparent direction, practically. The distances of some stars are so vast that the directions of one from all points of the earth's orbit are about the same. And in two intervals, separated by six months, when the earth is moving with equal velocities in opposite directions, α will have nearly equal values, and the displacements from the true position will be equal and opposite. The greatest value of δ for any star is only about one-third of a minute of angular measure.

**Fizeau's Experiment.**—Suppose a source of light to send a pencil to a plane mirror at a great distance placed normally to the path of the pencil. Imagine a toothed wheel near the source, and so placed that the light goes and returns by the space between two teeth. Now, the wheel may be rotated at such a speed that the light goes to the mirror through an aperture, and on its return is intercepted by the next tooth ; or, again, the wheel may be rotated faster, so that the light returns by the next aperture. On continuing to increase the speed, successive disappearances and reappearances would be observed.

This is the principle of Fizeau's method. The experiment was carried out in the following manner :—

At two stations, a considerable distance apart, are two telescopes, A B and C D. In Fizeau's experiment this distance was nearly three miles. The two telescopes are directed towards each other, so that light from the focus of the object-glass of either may enter the other. This is arranged by setting each telescope so that the image of the object-glass of the other is brought to the centre of its field of view. The eye-piece D is removed, and a plane mirror, M, is set at the focus of C, and at right angles to the axis of C, so that the rays entering C may return to B. It is clear that extremely accurate setting of M is not necessary, so long as the focus of C is in M. S is a source of light of which an image is formed at F, the focus of B, by means of a system of lenses and a plate of unsilvered glass, G, inclined at 45° to the axis of B. W is a toothed wheel, whose teeth pass over F, and which can be rotated at a known rate by means of clockwork and counting-wheels. When the wheel rotates slowly, on looking through A an image of S will be seen, for the light will go through two teeth to M, and return through the same space. As the speed of W is increased, the light will be cut off; and on still further increasing it, the light will reappear, because it goes to M through one



FIG. 129.

aperture and returns through the next; and so on. If, then, the speed of W giving complete extinction of the light is known, and the number of the extinction (whether it is the first second, third, etc.), the angle through which the wheel
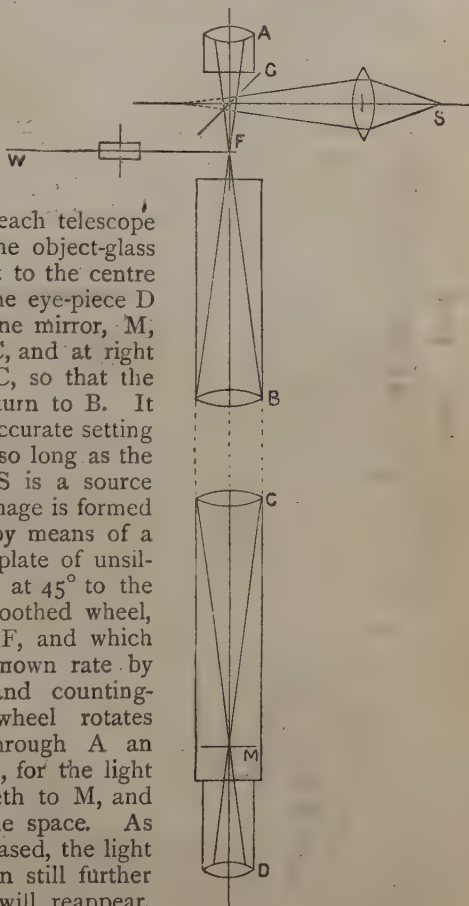
has turned between letting the light through and cutting it off again is known ; and thus can be found the time the light has taken to travel from F to M and back again ; and so the velocity of light in air.

The chief objection to this method is that the exact speed of the wheel at which complete extinction of the light takes place cannot be directly determined ; extinction appears to be complete at any speed which allows too small a quantity of light to pass to affect the eye.

Cornu has introduced improvements into this method. The speed of rotation of the wheel W was allowed to vary, its revolutions being registered electrically, so that its speed at any instant could be deduced. At the same time, the observer can, by means of a key, register any instant at which he desires to know the speed. Thus, by allowing the speed to vary continuously, and registering the instants at which the light disappears and reappears, the speeds at these two instants can be determined ; and the speed for complete extinction is the mean between them. Cornu has found by this method for the velocity of light in air, 300,330,000 ; and *in vacuo*, 300,400,000 metres per second.

**Foucault's Method.**—The figure shows a horizontal plan of Foucault's arrangement. Sunlight is directed through a
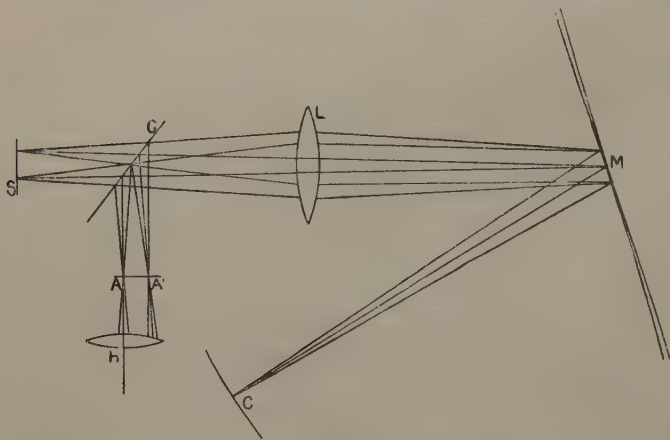


FIG. 130.

rectangular aperture, S, in the middle of which is a fine vertical wire. The light passes through a vertical plate of glass, G,

inclined at 45° to its path, and through an achromatic lens, L. At M is a small plane mirror which can be rotated about a vertical axis. At C is a concave mirror having its centre of curvature on the axis of rotation of M. The lens L is so adjusted that when M is suitably directed, an image of S is formed at C. Supposing M to be stationary, the light reflected back from C will form an image of S, which may be seen by reflexion from G, by means of a lens H; at A, say. Now, suppose M to be rotated with a moderate degree of rapidity. On account of the persistence of impressions, a continuous image of S will appear to be formed at A. The brightness of this image will depend upon the extent of the mirror C, for light is only being reflected back to L in the small portion of M's rotation in which it is sending light to C.

Suppose, now, that M is rotated very rapidly. During the time that light travels from M to C and back to M, M has turned through a small angle, and thus the light is reflected off from M by a slightly different path from that by which it arrived. This is true for each separate ray of light travelling from M to C; and whatever be the extent of C, all the rays on coming back to M are reflected back along paths inclined to their original paths at a common angle, equal to twice the angle through which M has turned during the passage of the light from M to C and back to M. This will cause the image of S formed by reflected rays, and seen by reflexion at the glass G, to appear displaced from A; to A', say.

Suppose that $a$ denotes the distance S M; $b$, the distance M C; $d$, the displacement A A' of the image of S; $n$, the number of turns made by M per second; V, the velocity of light. The time taken by light to travel to distance M C M is $2\frac{b}{V}$. In this time the mirror M turns through the angle $2\frac{b}{V} \times 2\pi n$ $= 4\pi n \frac{b}{V}$. Thus the deflexion of the rays reflected from M is $8\pi n \frac{b}{V}$. And $d$ being small, this is also very nearly $\frac{d}{a}$. Thus we get—

$$\frac{d}{a} = \frac{8\pi n b}{V}.$$

$$V = \frac{8\pi n a b}{d}.$$

In Foucault's later experiments the distance M C was made

large by causing the light to undergo several reflexions between the mirrors M and C; the distance thus obtained was 20 metres.

The mirror M was carried on a well-made axis, and very carefully balanced. It was driven by a blast of air acting on a sort of siren to which it was attached. It was coated with silver on the reflecting side. It was in some experiments driven at a speed of some 800 revolutions per second.

Foucalt employed this apparatus to determine whether the velocity of light in air or in water is the greater. For this purpose another concave mirror, D, like C, is added to the apparatus, and a tube of water placed between it and M, so that the light had to traverse the water in its passage from M to D and back again. This would cause the light to come to a focus further off than D; and to correct for this, a convex lens had to be used along with the tube of water to focus the light on D. Each of the mirrors C and D would now, when M is rotating, give rise to an image of S by reflexion at G; and when M is rotating slowly, these images would be superposed at A. Now, suppose that M is rotated very rapidly. It was found that the image produced by C, by means of rays that had passed entirely through air, was less displaced than that produced by D. It follows that the velocity of light in air is greater than that in water.

This experiment leads us to an important conclusion with regard to the nature of light. We shall see that the emission theory requires that the velocity, in a medium of greater refrangibility, shall be greater than in one of less refrangibility; but the undulatory theory requires that it shall be less. Now, water has a greater refrangibility than air; and we see that the velocity in it is less than in air. This experiment, then, shows that the emission theory, as it will be described, is untenable. Regarded as a crucial experiment to decide between the two theories, it decides in favour of the wave theory.

Captain A. A. Michelson, of the United States, has used the method of Foucault, modified in some respects, in a careful determination of the velocity of light. The distances between the aperture S and the revolving mirror M, and between this and the fixed mirror C, were made large—30 and 2000 feet respectively. The mirror C was plane, and the lens, which was of 150 feet focal length, was placed between it and M. By using the lens in this way, an image of considerable brightness could be obtained, although the distance

M C was made so large. If the mirror C is of the same breadth as the lens, and M is at the principal focus of the lens, then, as long as light from M falls on any part of the lens, it will reach C and return to M. In the experiment, in order to increase the distance S M, M was placed about 15 feet within the focus of the lens, and it was found that the light in this way reaching C gave a bright enough image when reflected back to S; although the light passing through the lens is then somewhat divergent, and all of it does not reach the mirror C.

In this, as in Foucault's experiment, it is necessary that an image of S should be formed exactly coinciding with S, when M is stationary. Now, L and M form an image of S, say $I_1$. An image of $I_1$—say $I_2$—is formed by C; and $I_2$ acts as the source of the reflected light. For this to produce an image coinciding with S, it is necessary for $I_1$ and $I_2$ to coincide. Thus $I_1$ must be formed on the surface of C. Then, whatever be the shape of C, $I_2$ coincides with $I_1$.

The light returning to S was not reflected to the side, as in Foucault's experiment. This was unnecessary, as a large deflexion was produced. An eye-piece, which can be moved laterally along a scale which measures its motion, is placed so that it can be set behind the slit S, that is, on the side of S of the incident light, or it can be moved away to observe the deflected image of S when the mirror is spinning round. The difference of the readings on the scale when the eye-piece is focussed on S and on its image is the amount of the deflexion.

The mirror M was rotated by a sort of air turbine, the driving pressure being susceptible of nice regulation. Its speed was about 256 revolutions per second. This was measured by comparing it with a tuning-fork kept vibrating by electrical means. To the tuning-fork was attached a steel mirror, and the light from M falling on this is reflected to a plate of glass in the eye-piece inclined to its axis, and then to the eye. The fork made about 128 complete vibrations per second, its exact rate being determined by comparison, by the acoustical method of beats, with a standard fork of about an octave higher. Now, when the mirror rotates and the fork vibrates, the successive flashes of light from the mirror M will generally fall on the mirror of the fork when this is in different positions, and a spread-out image is seen in the eye-piece. When the mirror rotates in the same time as the fork vibrates, a single distinct image is seen; if twice as quickly, two images are seen, and so on. The speed of the mirror was regulated by adjusting the pressure while observing the reflected images

from the fork; and, when a steady speed was got, an observation of the displacement of the image was taken. A displacement of about 133 mm. as against $\frac{7}{10}$ mm. in Foucault's experiment, was obtained.

The result obtained for the velocity by these experiments was 299,944,000 ± 50,000 metres per second, *in vacuo.*

The question has been considered whether lights of various colours travel with the same or different velocities; and some observers have thought that they had detected differences in the velocities; but the weight of evidence is in favour of the velocities of lights of all colours, that is, of all wave-lengths, whether in air or in empty space, being the same.

Immediately after the eclipse of a star, if lights of different colours travelled with different velocities, the light of some colours would reach us before that of others, and the star would appear coloured.

On account of aberration, the lights of various colours would reach us along different directions, and the star would appear like a narrow spectrum, or, at least, would present coloured edges.

In Fizeau's experiment suppose, for instance, that the blue light travels faster than the red. Then as the speed of the rotating wheel is increased so as to begin to cut off the light, the red light is cut off first, and the image appears blue. When the eclipse is over, and the image begins to reappear, the red light begins to appear first, and the image appears red.

In Foucault's experiment the image of the aperture would consist of a series of coloured images having different positions. Thus the appearance would be that of a spectrum, or, at least, of an image having coloured edges.

It has not been proved that any of these appearances occur. Hence we infer that there is no appreciable difference, in air or *in vacuo*, between the velocities of lights of various colours.

## CHAPTER X.

### *WAVE-MOTION.*

IMAGINE a particle to be moving to and fro along a straight line A B, performing oscillations under the action of any forces, about the point O, which we may suppose to be the position of the particle when at rest, or its equilibrium position.

Now, suppose that the motion of the particle is such that it repeats itself ; so that, after having performed a certain motion in a definite interval of time, the particle always begins again and executes precisely the same motion in the same period ;

A    O    B

FIG. 131.

that is, at any instant the particle is just where it was at a definite interval of time before (this interval being the same, at whatever instant of the particle's motion we are considering it), and is moving in the same direction and with the same velocity. The motion of the particle is then said to be **periodic**. And the interval of time taken by the particle to execute its complete motion before beginning again, or the interval between two corresponding instants at which the particle is in the same position and moving in the same way, is called the **period** of the motion.

The simplest sort of periodic motion from a dynamical point of view is simple harmonic motion. This is frequently written, for short, S.H.M. We shall now define it.

Suppose a point, P, to move round the circumference of a circle with uniform velocity. Let Q be the foot of the perpendicular let fall from P on a fixed diameter, A A′, of the circle. The motion of Q, to and fro along A A′, is called simple harmonic motion. O A is called the amplitude of the motion.
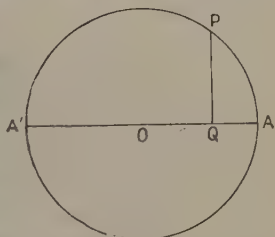
This motion is approximately illustrated by the motion of one end of a rod, this end being guided so as to move in a straight line through the centre of a wheel, and the other

FIG. 132.

end being carried round with uniform velocity by the wheel. If the rod is so long that it may be supposed to undergo no appreciable angular displacement, the linear displacements of its two ends along the line in question are appreciably equal, and the motion of the end which moves along this line is appreciably S.H.M.

Suppose that, in the above figure, the radius OA = $a$, and the angular velocity of the point P = $\omega$; P performs a revolution in time—

$$T = \frac{2\pi}{w}$$

This is, then, the period of the motion of Q. The

acceleration of P is along P O, and is $\omega^2 a$. The acceleration of Q is the component of this along A O; that is, it is $\omega^2 \cdot$ QO.

Thus, if the point Q is performing S.H.M. with O as mean position, the acceleration of Q is always directed to O, and proportional to the displacement of Q from its mean position.

Let OQ = $x$, and let the acceleration of Q towards O be $kx$. Or, supposing distances, accelerations, etc., from O to the right to be positive, and those to the left negative, we may call the acceleration of Q simply $-kx$, $k$ being a positive quantity. The acceleration of P would now be $ka$, and so $\sqrt{k}$ is the angular velocity of P; and the period of a complete revolution of P is $\dfrac{2\pi}{\sqrt{k}}$. This is, therefore, also the period of Q's motion.

Thus, when the acceleration of Q at any point is $-kx$, the period of Q's motion is $\dfrac{2\pi}{\sqrt{k}}$

Or, we may write for the period of a S.H.M.—

$$2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}} ;$$

the numerical magnitude of the quotient under the radical being, of course, understood.

It is easy to see that if a point or a particle is moving in a straight line in such a manner that its acceleration is always *towards* a given point in the line, and is proportional to its displacement from that point, it performs S.H.M. about that point as mean position. For we may imagine another particle performing S.H.M. so that its acceleration in any position is the same as that of the given particle in the same position, and its motion is of such amplitude that its velocity, in passing through its mean position, is the same as that of the given particle at the same position. Then, if we suppose that the two are moving through the mean position at the same instant and in the same direction, they will always keep together; that is, the given particle will describe S.H.M.

The velocity, $v$, with which Q passes through its mean position, is equal to the velocity of P, $a\omega$.

The complete to-and-fro motion of Q is sometimes called an *oscillation* or a *vibration;* $\dfrac{2\pi}{\sqrt{k}}$ is called the **vibration period.**

The **phase** of a vibrating particle denotes the point at which the particle is in its motion at any instant. It is generally taken to mean the fraction of a complete period that has elapsed since the particle was last in its mean position moving through it in the sense reckoned positive. Thus the phases of Q at A, A' are $\frac{1}{4}$ and $\frac{3}{4}$, and at O it is o or $\frac{1}{2}$, according as Q is moving to right or left, if we agree to consider distances measured from O to the right as positive.

Thus if T is the period, and $t$ is the time since the particle was last moving through its mean position in the positive sense, its phase is $\frac{t}{T}$. It should be noticed that sometimes $t$ is called the phase, and sometimes $\frac{2\pi t}{T}$.

Imagine a point to start from O, and to begin to move to the right. At the end of a time, $t$, let Q be its position.

Then $\angle\ B'OP = 2\pi\dfrac{t}{T}$.

Thus the displacement of Q is—

$$a \sin \frac{2\pi t}{T}.$$
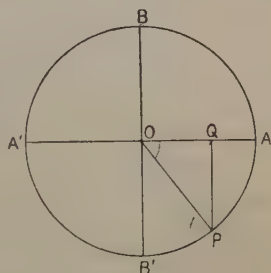
FIG. 133.

This expression is general, having a negative value when Q is on the left of O.

The velocity of Q at the same instant is the resolved part along O A of P's velocity. Thus it is $a\omega \cos B'OP$ ; that is—

$$\frac{2\pi a}{T} \cos \frac{2\pi t}{T}.$$

If the time $t$ is measured from the instant at which the phase of Q has the value $\dfrac{\epsilon}{2\pi}$, then the expressions for the displacement and velocity of Q are—

$$a \sin \left(2\pi\frac{t}{T} + \epsilon\right)$$

and—

$$\frac{2\pi a}{T} \cos \left(2\pi\frac{t}{T} + \epsilon\right).$$

Suppose a particle has two independent simple harmonic motions in the same straight line, and of the same period. Denote the displacements due to these by—

$$a_1 \sin \left(2\pi\frac{t}{T} - \epsilon_1\right)$$

and—

$$a_2 \sin \left(2\pi\frac{t}{T} - \epsilon_2\right).$$

The resulting displacement, the sum of these, is—

$$(a_1 \cos \epsilon_1 + a_2 \cos \epsilon_2)\sin \frac{2\pi t}{T} - (a_1 \sin \epsilon_1 + a_2 \sin \epsilon_2) \cos \frac{2\pi t}{T}.$$

This may be written—

$$a \sin \left(2\pi\frac{t}{T} - \epsilon\right),$$

if $a \cos \epsilon = a_1 \cos \epsilon_1 + a_2 \cos \epsilon_2$,
and $a \sin \epsilon = a_1 \sin \epsilon_1 + a_2 \sin \epsilon_2$.

Thus $a^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos (\epsilon_1 - \epsilon_2)$,

and $\tan \epsilon = \dfrac{a_1 \sin \epsilon_1 + a_2 \sin \epsilon_2}{a_1 \cos \epsilon_1 + a_2 \cos \epsilon_2}$.

A geometrical construction may be made for the resulting
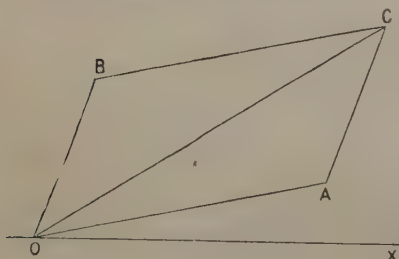


FIG. 134.

S.H.M. For from O draw O A, O B equal to $a_1$ and $a_2$, and making the angles X O A and X O B equal to $\epsilon_1$ and $\epsilon_2$. And construct the parallelogram O A C B. Then, since the projection of O C on O X is equal to the sum of the projections of O A, O B, we have—

$$OC \cos XOC = a_1 \cos \epsilon_1 + a_2 \cos \epsilon_2.$$

And similarly—

$$OC \sin XOC = a_1 \sin \epsilon_1 + a_2 \sin \epsilon_2.$$

Thus $OC = a$; and $XOC = \epsilon$.

In a similar way we may compound any number of S.H.M.'s of equal period and different phases by starting from any point and drawing a polygon whose sides shall be equal to the amplitudes of the S.H.M.'s, and whose inclinations to any straight line, when multiplied by $\dfrac{T}{2\pi}$, shall be equal to the phases of the motions ; or whose inclinations are the **phase-angles** of the motions, as $\epsilon_1, \epsilon_2, \ldots$ may be called.

Suppose we have a series of particles having mean positions equally spaced out along a straight line, and performing S.H.M.'s about these mean positions in paths at right angles to the given straight line, and all in one plane, the amplitudes and the periods of the motions being the same for all, but the phase falling off uniformly from particle
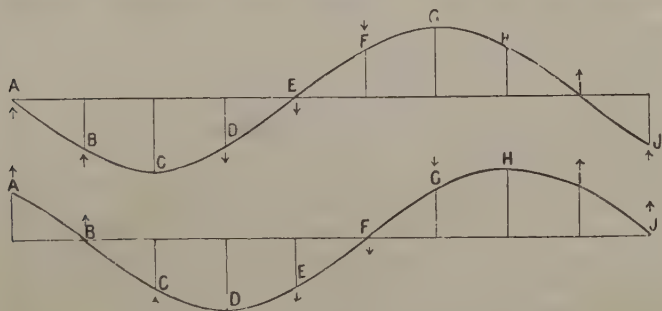


FIG. 135.

to particle. Let, for example, the particles A, B, C, . . . be moving in such a manner, each lagging in phase by $\frac{1}{8}$ behind the one to the left of it. Consider the grouping of the particles at any instant. Suppose A to be moving upwards through its mean position. B is moving up towards its mean position. C is at its greatest depth, and not moving at the instant. D is moving down from its mean position. E is moving down through its mean position. And so on.

Consider, again, what the grouping is when $\frac{1}{8}$ of a period has elapsed (see lower half of Fig. 135). Each particle is now in the phase in which the preceding one was in the first case. The grouping is the same as before, but the particles do not occupy the same positions in it. The figure representing it is displaced to the right by a distance equal to that between two consecutive mean positions.

Continuing to represent the successive arrangements of the particles in this way, we see that the figure representing the grouping moves to the right at a uniform rate. At the end of a whole period each particle will just have got back to the condition in which it was at the beginning, and the grouping will be just the same. The figure of grouping will have travelled in this time through the distance A I, that is, through the distance from any particle to the next one in the series, which is always in the same phase as it.

We have thus a series of *waves*, transverse to the line of particles, passing along them. A particle which is at any instant at its greatest height above the horizontal line is at a wave-*crest ;* and a particle at its greatest depth is at a *trough.*

The **wave-length** is the distance between two consecutive crests or two consecutive troughs, or the distance from any particle to the next one which is in the same phase. The **velocity of propagation** of the waves is the velocity with which the crests or the troughs move along, or the velocity with which any given phase is propagated. Let T denote the period of vibration of a particle, $\lambda$ the wave-length, and V the velocity of propagation of the waves. Then, in the time T, each particle goes through its complete motion and comes back to its old position ; or, where there was a crest, or any given state of displacement of the particles at the beginning of the time T, there is again the same state at the end of that time. Therefore in this time the waves have travelled through the distance $\lambda$. And they travel with velocity V. Thus we have the important relation—

$$\lambda = \text{VT}.$$

Suppose, now, that the series of particles along which the waves pass is indefinite in number. Supposing, as before, that the period and amplitude are the same for all, being equal to T and $a$, and that the phase falls off uniformly along the line of particles ; that is, at any instant the difference of phase between two particles is proportional to the distance between them. The difference in phase between any particle, say the particle A, and one, P, at a distance, $x$, from it, is $\dfrac{x}{\lambda}$. Measure the time $t$ from an instant when the phase of A is o. The displacement of A at time $t$ is—

$$a \sin 2\pi \frac{t}{\text{T}}.$$

The time at which the phase of P is o is $\dfrac{\text{T}x}{\lambda}$. Thus at time $t$, since the interval that has elapsed from this instant is $t - \dfrac{\text{T}x}{\lambda}$, the displacement of P is—

$$a \sin 2\pi \left( \frac{t}{\text{T}} - \frac{x}{\lambda} \right).$$

remains unchanged. Suppose its entire velocities in the first and second media are $v, v'$; and $\mu$ the index of refraction from the first to the second; $i$ and $r$ the angles of incidence and refraction. Equating the velocities parallel to the surface, we get—

$$v \sin i = v' \sin r;$$
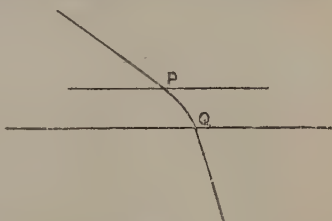$$\therefore \mu = \frac{v'}{v}.$$

In the same way, refraction with an index less than unity, as from glass to air, would be

FIG. 137.

explained by supposing that the particle is repelled by the surface, but not sufficiently to prevent it from entering the second medium.

Let us next consider how reflexion and refraction may be explained according to the wave theory. We must first, however, consider the propagation of a series of waves in space. We have considered the passage of waves along a line of particles, such as the particles of a stretched string. But we may have a series of waves passing through a solid body, and affecting each particle of it, or within a certain portion of it, with periodic disturbances. And, in the same way, we may have a series of waves passing through space, and affecting the ether. Now, the ether may be so circumstanced that the waves pass through it with the same velocity everywhere and in all directions, as when the space contains air everywhere; or the velocities may be different at different places, as when part of the space contains air, and another part some refracting medium, such as glass.

Any surface which is such that at any instant the phase of disturbance is the same at any point of it, is called a *wave-front*. Thus if waves are radiated into a homogeneous medium from a single point, in such a manner that the disturbance starts in the same phase along any line from the point, the wave-fronts will be spheres having the point as centre. A very small portion of such a sphere, as compared with its distance from the centre of the disturbance, may be regarded as a *plane wave-front*.

To consider more fully, from a mechanical point of view, the passage of a series of plane waves through a homogeneous medium. Imagine a single plane disturbance. When this has reached the indefinitely extended plane A B, each point of

this plane may now be considered as an origin of disturbance, and would, if acting alone, send out a spherical disturbance into the medium. At the end of a given time after reaching A B, suppose the disturbance has travelled in any direction through a distance *r*.

Imagine spheres described having all the points of A B as centres, and each with radius *r*. These will all touch, and have



FIG. 138

as envelope the plane C D, parallel to A B, and at a distance *r* from it. The disturbance will, therefore, in this time, have reached this plane. Let P be a point at a distance from A B less than *r*. The disturbances reaching P at the instant in question are from the centres of spheres on which P lies, and they thus come to P along paths oblique to A B. In the case of a uniform succession of waves, we shall see that the effect thus produced at P is very small as compared with that produced at a point on the wave-front C D. And the effect at any point is practically confined to the disturbances sent to it from a very small element of A B surrounding the foot of the perpendicular from the point to A B. A series of plane-waves thus travels in a direction at right angles to the wave-fronts.
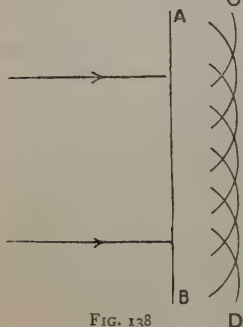
In the same way, with waves having wave-fronts of any form whatever, the direction in which the disturbances are travelling at any point is normal to the wave-front at that point.

**Reflexion of Plane Wave.**—Suppose a plane wave, with wave-front A B, to meet the plane reflecting surface X Y. We must suppose the surface to act in the following way : When a disturbance in the ether reaches any point of it, that point becomes immediately a centre of disturbances which it sends back into the medium. This may be illustrated by means of a long chain hanging from a fixed point. A wave may be sent up the chain so as to leave the parts of the chain over which it has passed at rest. When the wave reaches the top and can go no further, it is reflected, and travels down the chain again.

Let the plane of the figure contain the normal to the reflecting surface and the normal to the wave-front, or the direction of the incident light. As the various points of the wave-front reach the surface, they become centres of disturbances

which are reflected, and by the time that B has reached the surface the disturbance sent out from A has reached a distance from A equal to B C. The reflected disturbance is in a plane which touches a series of spheres with centres from A to C, and having radii which fall off uniformly from A D (equal to
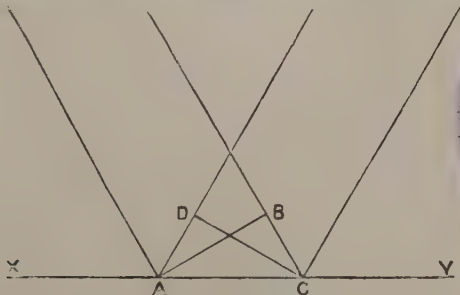


Fig. 139.

B C) to zero. Any line in the wave front parallel to the surface, and therefore at right angles to the plane of the figure, will all reach the surface at the same instant, and give rise to a set of spheres of disturbance always having the same magnitude. Thus the reflected wave will be such that C D is its trace in the plane of the figure; it will be at right angles to the plane of the figure, and make an angle with X Y equal to that which A B makes.

It is clear that the direction A D, in which the reflected wave travels, obeys both the laws of reflexion.

**Refraction of Plane Wave.**—Suppose a plane wave, with wave-front A B, to meet the plane surface X Y of a refracting medium. As before, suppose that the plane of the figure is at right angles to X Y and to A B. Suppose that the disturbances on entering the medium have a different velocity from that which they had before. Suppose the velocities before and after entering the medium to be $v$ and $v'$. By the time that B has reached the surface X Y, the disturbance from A has travelled a distance A D, such that $AD : AB = v' : v$. And the entire wave-front in the medium is the plane touching a series of spheres having centres from A to C, and with radii uniformly falling off from A D to zero, and such that all those with centres on a line at right angles to the plane of the figure are equal. We shall then have the directions in which the incident and refracted waves travel in the same plane with the normal to the surface X Y : and $\sin BAC : \sin DCA = v' : v'$.

But B A C, D C A are the angles of incidence and refraction. Thus the two laws of refraction are obeyed.
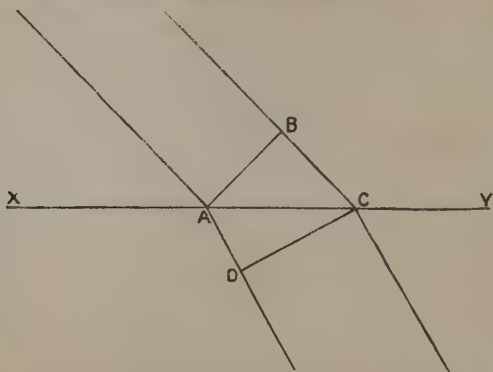


FIG. 140:

We have now a crucial test between the emission and the undulatory theory. In passing from one medium to another in which the velocity of light is less, according to the emission theory the index of refraction should be less than unity; according to the undulatory theory it should be greater than unity. Now, direct experiments on the velocity of light in various media, and on refractive indices, show that it is the undulatory theory that gives the correct explanation.

**Rectilinear Propagation of Light.**—The emission theory readily accounts for the rectilinear propagation of light. This was one of the greatest difficulties in the way of the undulatory theory. But this theory can satisfactorily, although not easily, account for this behaviour of light.

Let O be a visible point radiating disturbances into
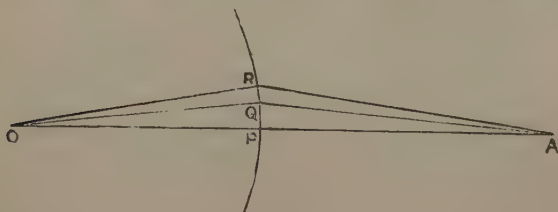


FIG. 141.

surrounding space. Let A be a point at a distance $a$ from O. Suppose a series of periodic waves of length $\lambda$ to be sent out

from O. Let the sphere P Q R denote a wave-front of the disturbances sent from O, so that at any instant every point of P Q R is in the same phase of disturbance. We may then regard all the points of this phase as acting as secondary origins of disturbance which are sent to the point. But because of the different distances of these points from A, a given phase of disturbance starting from these points does not reach A all at the same time. And the disturbance at A at any instant is made up of disturbances from the various points of P Q R, the actions of which at A, to a great extent, destroy each other. Suppose, for instance, that two points are sending to A waves of S.H.M. of the same amplitude. And let the two points be at distances from A differing by half a wave-length. The resulting disturbances at A will always just destroy each other, or they will produce complete interference at A. Now, the disturbances from the various points of P Q R will not produce complete interference at A; but they will interfere to a great extent. The way in which the resultant disturbances are produced at A may be approximately determined by the following simple method : Let P be in a straight line with O and A, so that P A is normal to P Q R. Divide the sphere into a series of zones, P Q, Q R, etc., so

that $AQ - AP = AR - AQ = \ldots = \dfrac{\lambda}{2}$. The disturbances

from the various points of any one of these elements may be regarded as assisting each other's action at A.

At certain instants the disturbances at A due to the various points of such an element are all in the same sense, those due to the bounding points just vanishing. This is the maximum disturbance produced at A by the zone, and if this were either enlarged or diminished, this maximum disturbance at A would be less. The first, third, fifth, etc., zones act together to produce disturbance at A. The others oppose the action of these.

These elementary zones into which the wave-front is thus divided are called *half-period elements.*

Let us next consider the sizes of these zones. Let OP = x ; PA = y ; x + y = a. Let PQ = s. To find AQ to the second order of small quantities.

$$AQ^2 = AO^2 + OQ^2 - 2AO \cdot OQ \cos AOQ$$

$$= a^2 + x^2 - 2ax \left( 1 - \frac{s^2}{2x^2} \right)$$

$$= (a - x)^2 + \frac{as^2}{x}$$

$$= y^2 + \frac{as^2}{x};$$

$$\therefore AQ = y + \frac{as^2}{2xy}.$$

$$AQ - AP = \frac{\lambda}{2};$$

$$\therefore \frac{as^2}{2xy} = \frac{\lambda}{2}.$$

$$PQ^2 = \frac{xy\lambda}{a}.$$

$$\text{So } PR^2 = \frac{2xy\lambda}{a}.$$

And so on.

Thus the areas of the first few elementary zones, considered as lying in one plane, are $\pi PQ^2$, $\pi(PR^2 - PQ^2)$, and so on; that is, each is equal to $\frac{\pi xy\lambda}{a}$.

The zones are thus, for some distance round P, of about equal area. But, on account of their increasing distances from A, and on account of the increasing obliquity of their surfaces to the lines drawn from them to A, the disturbances which they send to A gradually fall off, starting from the first at P. Let the maximum disturbances produced by the zones at A be $m_1$, $m_2$, $m_3$, . . . Then the resultant disturbance at A is—

$$m_1 - m_2 + m_3 - \ldots$$

We must suppose that these terms rapidly diminish. Thus the effect produced at A is due to a small portion only of the wave-front round P. A screen placed anywhere in the region, but so as not to come between O and A, or very near to the line O A, will produce no effect at A. Thus the light from O which reaches A may be said to travel in a straight line to A.

If light of a definite quality, that is, having a definite wave-length and a definite vibration-period in air, passes from one medium to another, since the velocity is changed there must be a change in the wave-length or in the vibration-period of a particle in the medium. Now, all the vibrations reaching the boundary-surface of the media in a given time are sent on into the second medium, and pass through any point in the path of the light in the second medium. Thus the vibration-

periods of particles in the two media must be equal. And, since velocity = wave-length ÷ period, it follows that if the velocity is changed in any ratio in passing from one medium to another, the wave-length is changed in the same ratio.

Suppose $v$ and $v'$ are the velocities in air, taken as standard medium, and some other medium of refrangibility $\mu$. Then $\mu = \frac{v}{v'}$. In the time taken by a wave to traverse a length $l$ of the medium $\mu$, a length of air would be traversed equal to $l \cdot \frac{v}{v'}$, or $l\mu$. This is called the **optical length** of the path in the medium $\mu$. The optical length of the path of a ray through several media in general means the length of path that would be traversed in air, taken as standard medium in the same time, and if $l_1$, $l_2$, . . . are the actual lengths in the media $\mu_1$, $\mu_2$, . . the optical length of the path is—

$$l_1\mu_1 + l_2\mu_2 + . .$$

The optical length of the path of a ray of light of given quality is clearly proportional to the number of wave-lengths in it. For the number of wave-lengths in path of length $l$ in medium $\mu$ is proportional to $l$, and inversely proportional to the velocity in the medium, that is, directly proportional to $\mu$. Thus it is proportional to $l\mu$.

## CHAPTER XII.

### *INTERFERENCE OF LIGHT.*

IMAGINE two point-sources of light, A and B (Fig. 142), very near each other, and emitting light of a single wave-length, $\lambda$, the undulations of which always leave the sources in the same phase. Then at any point which is equidistant from A and B, the undulations which arrive are always in the same phase. And at a point, P, not equidistant from A and B, the undulations are in phases which differ by an amount depending on the difference of the distances A P, B P, so long as A and B go on emitting undulations uniformly. If after A and B have emitted a series of waves there is a break, and then they emit another series following directly on the first, but not regularly connected with it, so that the interval between the last vibration of the first series and the first of the second is the

same as that between two successive vibrations of one series, then, while the vibrations of the first series from one of the points and those of the second from the other point are going through P, it is manifest that the phase-difference will not depend alone on AP ~ BP, but on the nature of the break as well. If, however, AP ~ BP is very small, and A and B emit what may be considered long series of vibrations, then practically always the

A •
B •

F

FIG. 142.

phase-difference at P depends only on AP ~ BP. Suppose P is a point on a screen parallel to the line A B, and very far from A and B compared with the distance A B. Then, if the phases at P of the two series of waves are the same, these waves will reinforce each other at P, and the illumination will be greater than that due to one alone. If the phases are opposite, the disturbances at P at any instant due to the two sets of waves will be in opposite directions, and the lights will tend to destroy each other. In this case, if the sources are of equal brightness, P being nearly equidistant from them, there should be practically absence of illumination at P.

If, then, AP ~ BP is zero or a multiple of $\lambda$, there is brightness at P. If AP ~ BP is an odd multiple of $\frac{\lambda}{2}$, there is darkness at P. And at points between two such points there is continuous change of intensity of illumination.
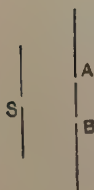
C

Young performed an experiment of which the theory has just been indicated. Sunlight was admitted into a darkened room through a slit, S, and fell on a screen containing two pin-holes, A, B, very near to each other. Thus were produced two sources emitting vibrations identical in phase. At some distance from this was a screen, C D. On this screen was seen

S

A
B

FIG. 143.

D

a series of coloured bands in the direction at right angles to
A B. As the distance between the pin-holes was increased, the
bands diminished in width, and finally disappeared.

Let us next consider more exactly the positions in which
these bright and dark **bands,** or **interference fringes,** will be
formed. Suppose at first the light to be simple, or all of a
single wave-length.

Let $d$ = the distance between two very near sources, A
and B, emitting identical vibrations ; $\lambda$ = the wave-length of
the vibrations ; $a$ = the distance from A or B to the screen.

From the point O, midway between A and B, let fall O M
perpendicular to the screen. Take a point P in plane A B M,
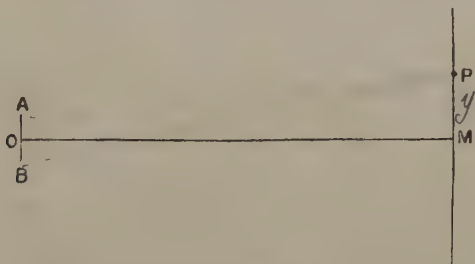


FIG. 144.

at distance $y$ from M, $y$ being small compared with O M.
Then, to calculate the value of $BP - AP$ approximately, we
have—

$$AP^2 = OM^2 + (PM - OA)^2$$

$$= a^2 + \left( y - \frac{d}{2} \right)^2 ;$$

$$BP^2 = a^2 + \left( y + \frac{d}{2} \right)^2 :$$

$$\therefore BP^2 - AP^2 = 2dy ,$$

$$BP - AP = \frac{2dy}{BP + AP} = \frac{dy}{a} \text{ very nearly,}$$

omitting small quantities of the second order. For since A P
is the hypothenuse of a right-angled triangle, of which one
side is $a$ and the other is very small, A P differs from $a$ by
a small quantity of the second order. Similarly for B P.

According as $BP - AP$ is an even or odd multiple of $\dfrac{\lambda}{2}$,

there will be brightness or darkness at P. Put $BP - AP = n\frac{\lambda}{2}$. Then we have—

$$dy = an\frac{\lambda}{2}.$$

And the distance from bright to bright or from dark to dark is the amount by which $y$ varies as $n$ varies by $2$; that is, it is—

$$\frac{a\lambda}{d}.$$

At the point M, equidistant from A and B, there is brightness, and at points on a line through M at right angles to the plane A B M. In the same way, if P is a point such as we have considered, satisfying the condition $BP - AP = $ constant, points on the screen and close to P satisfying the same condition are practically on a straight line through P. Thus we have a series of bright and dark bands on the screen.

Other instances occur in physics by which the subject of interference may be illustrated. Suppose, for instance, that two tuning-forks of the same pitch are sounding with the same intensity. This denotes that they are sending vibrations through the air of equal frequency, and, for points at equal distances from them, of equal amplitudes. Interference of the sounds emitted would then be produced if the forks be set at a distance apart, great as compared with the wavelength of their vibrations. That is, at certain points the vibrations would be opposed, and at others they would assist each other. The wave-length of a note is, however, so great—several feet, it may be—that the interference of sounds is more readily shown by another method. Let the forks be set close together, and be made slightly out of tune with each other, which may be done by loading a prong of one a little. If they are now sounded, they will, at regular intervals, be tending to produce, at the same place, vibrations of the air-particles in the same and in opposite directions, as the quicker fork is first in pace with the other, then catches it up by half a period, then gets into pace with it again, and so on. Thus the acoustical phenomenon of "beats" will be observed, that is, regularly alternating increase and diminution of the sound heard.

Instead of two point-sources, A and B, two line-sources of light may be used with advantage, if in these any two points at the ends of a straight line at right angles to the

line-sources are corresponding points emitting identical vibrations. For then each such pair of points will produce its bands on the screen, and the bands produced by all such pairs are merely superposed. Thus a more brilliant effect is produced.

Suppose, next, that the light used is compound in quality, that is, having a variety of wave-lengths; say that it is white light. Then the light of each simple quality will produce its bands, there being always a white central band at M; but the distance between consecutive bright or dark bands varying, being proportional to $\lambda$. The effect produced on the screen, being the superposition of all these bands, is a mixed effect with a white central band. The other bands form a limited series with brightly coloured edges. This indicates that the lights of different wave-lengths (giving bands of different widths) are lights of different colours. This can be proved more clearly as follows: Use light of a definite colour, by interposing a coloured glass in the way of white light, or by taking the light from a part of a spectrum; and observe the breadth of the bands produced. On using light of a different colour, bands of a different breadth will be produced. It may be found in this way that the various colours of the spectrum all have their characteristic wave-lengths, red having the longest, and violet the shortest. From the formula given above, the wave-length for any given colour can be determined by this experiment. In the ordinary spectrum, as we pass up from red to violet, the wave-length becomes shorter and shorter. The wave-lengths of the extreme visible rays in the spectrum are about 0·00076 and 0·0004 mm.

Light of one definite wave-length, that is, of a definite spectral colour, is called *monochromatic*. Light which is very nearly, but not quite, monochromatic may be obtained from a sodium flame, that is, the flame of a Bunsen burner in which is a salt of sodium. Another method is to receive a spectrum on a diaphragm in which is a narrow slit, so that practically light of a definite wave-length passes through the slit. Here again the light is not absolutely monochromatic, for its wave-length varies from one side of the slit to the other.

We shall next consider some of the means used for obtaining interference phenomena, and the methods of examining them.

**Lloyd's Mirror.**—Lloyd used a narrow slit, S, and placed a mirror M so that S is parallel to it, and very close to its plane. Thus the light from S falls on M almost at grazing

incidence. A point P of the screen then receives light from S, and the image S′ of S formed in M. S and S′ act as the two sources, and may be brought very close together by suitably adjusting M. At O, where the plane of M meets
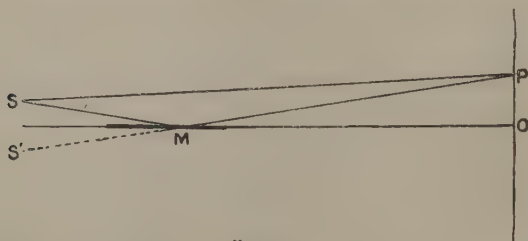


FIG. 145.

the screen, we should have the central bright band if the sources S and S′ sent out vibrations exactly in time with each other. But Lloyd found that a dark band is formed at O, all the bands being displaced by the distance between a dark and bright. He explained it by supposing that the phase of the reflected vibration is altered by one-half.

**Fresnel's Two Mirrors.**—Fresnel used two mirrors, O M, O N, meeting in a line at O, and inclined to each other at a very obtuse angle, nearly 180°. S is a source of light
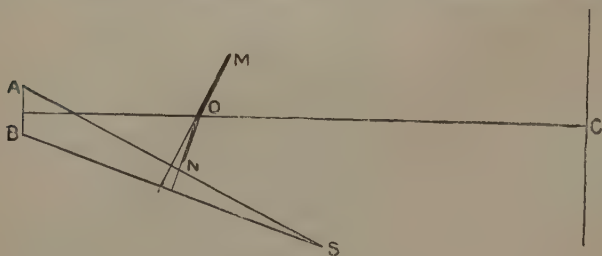


FIG. 146.

which produces images A and B in the mirrors, these images being very near to each other. Constructing A and B in the ordinary way, it is clear that A O and B O are each equal to S O. Thus the straight line from O to the middle point of A B is perpendicular to A B. Then if a screen is placed at C at right angles to this straight line, A and B will act as two identical sources, and produce a system of interference bands on it, the central bright one being at C.

**Fresnel's Bi-prism.**—This is a glass prism with two faces, making a very obtuse angle, nearly 180°, and the other two angles equal. A narrow slit is set at S parallel to the edges of the prism, and so that S A is at right angles
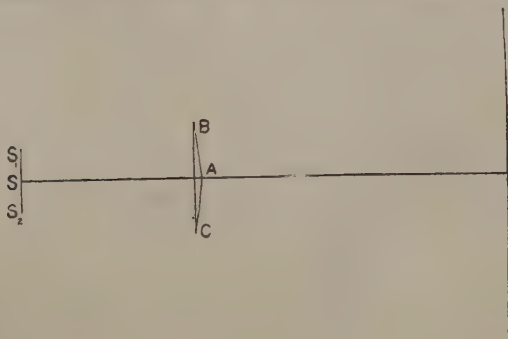


FIG. 147.

to B C, and is illuminated. Two images will be formed of S in the two halves of the prism. Let these be $S_1$, $S_2$. $S_1$, $S_2$ will be very near together. For suppose each of the angles at B and C is $a$, and that the refrangibility of the glass for the light used is $\mu$. Then $SS_1$ is practically at right angles to S A, and equal to $SA(\mu - 1)a$. So for $SS_2$. Thus the distance between $S_1$ and $S_2$ is $2SA(\mu - 1)a$. These two sources, $S_1$ and $S_2$, will then produce interference bands on a screen at right angles to S A produced.

Suppose mixed, say white, light to be used in this experi-
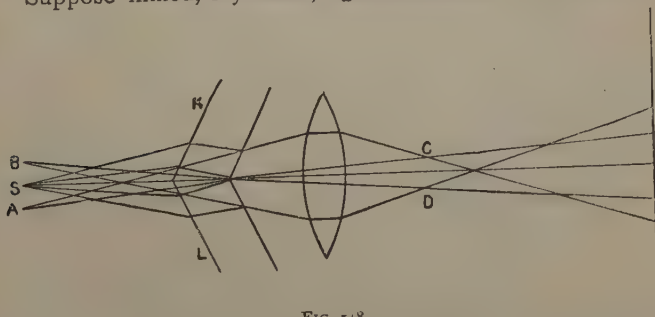


FIG. 148.

ment. The images of S of the various colours will be formed in slightly different positions, because of the dispersive power

of the glass. The blue images will be farther apart than the red. Thus, both on account of the red light having a longer wave-length, and the red images being closer together, the red bands will be farther apart than the blue. The superposition of the bands of various colours, or the effect produced by white light, will thus show more colouring in this case than with the two mirrors or with the single mirror.

The following methods have also been used for obtaining interference bands :—

Two glass plates (Fig. 148), K L (called *bi-plates*), cut from the same piece, are inclined at a very obtuse angle. They form two virtual images, A and B, of a source S. By means of a convex lens, real images, C and D, are formed of these, and C and D will produce interference bands on a screen suitably placed.

A convex lens is cut into halves ; and the two parts form two images, A and B, of a source S. A and B are then used



FIG. 149.

to produce interference bands. By bringing the two parts of the lens closer together, we can make A and B as near together as we like, and thus broaden the bands.

In all these experiments we have spoken of the means used as producing two identical sources of light, that is, two sources sending out, point for point, identical vibrations. We may, however, consider that the end to be attained is to lead the light from a source to a screen, so that that which reaches any point of the screen from any point of the source may reach it by two different paths. There must be, as a rule, a small difference in the lengths (or optical lengths) of the paths traversed, varying continuously from one point of the screen to another. This is what is done in the experiments mentioned.

**The Optical Bench or Bank.**—The optical bench used for experiments in interference of light and other experiments in physical optics is generally of more solid construction than that used for measurements in geometrical optics, such as focal lengths of lenses and mirrors. This is necessary

because, in the experiments that we have now to deal with, it is required to make measurements of very small quantities across the run of the bench; and want of rigidity in the bed, or of stability in the standards carrying the measuring apparatus, etc., would be fatal to such measurements. The bed of the bench is made of metal, about a metre in length, and is of sufficiently heavy construction to afford the necessary stability. The uprights or standards for carrying various pieces should also be solidly made, and capable of being moved along the bed without any side motion, and of being clamped in any desired position. A scale on the bench indicates the relative positions of the standards. An optical
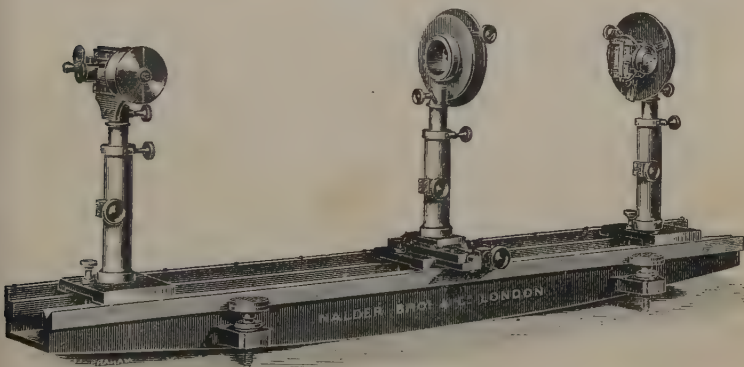


FIG. 150.

bench designed for measurement in physical optics is shown in the accompanying illustration.

**The Micrometer Eye-piece,** used with the bench for making observations on interference bands, etc., is an eye-piece, E, with a fine vertical cross-wire in its focus. It is capable of being moved in a frame in a direction at right angles to its optic axis. by means of a well-cut screw of known pitch; generally $\frac{1}{2}$ mm. pitch. There is a scale on the frame to measure the distance through which the eye-piece has been moved, and a divided head or drum, D, attached to the screw, measures the fraction of the traverse due to a whole turn. Thus with a $\frac{1}{2}$ mm. pitch the scale would be divided to $\frac{1}{2}$ mms., and if the drum were divided into fiftieths, the position of the eye-piece may be estimated to $\frac{1}{100}$ mm. The arrangement is mounted on a standard on the bench so that the optic

O

axis of the eye-piece is parallel to the bench, and the scale
along which it can be moved is horizontal and at right angles
to the bench.

Interference bands produced by any method may be
readily measured by means of the micrometer eye-piece.  To
measure the bands, for instance, produced by Fresnel's
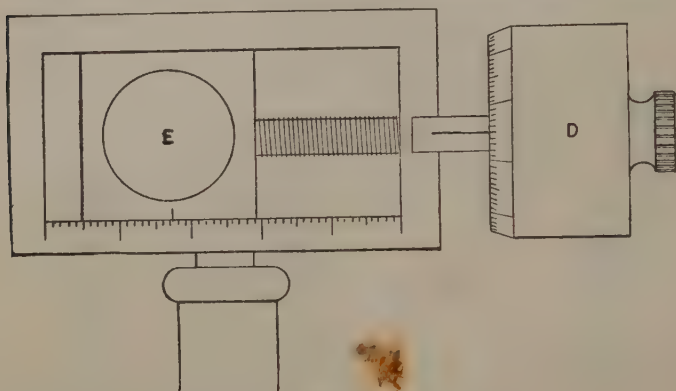


FIG. 151.

bi-prism, three standards are set on the bench, one at one
end to carry a vertical slit, another to carry the bi-prism, and
a third at the other end to carry the eye-piece.  Before the
slit is placed some source of light.

Suppose, for example, we wish to observe the interference
bands produced by sodium light, and to measure the wave-
length of the light.  A Bunsen burner with a sodium salt in
the flame would be used before the slit, and the bench set up
as described.  To get good bands the slit must not be too
wide, but wide enough to admit a fair quantity of light (the
best width would be found by trial), and it must be parallel
to the edges of the bi-prism.  This may be got by adjusting
the bi-prism while looking through the eye-piece till the bands
are at their best.  To measure the bands the run of the eye-
piece should be at right angles to them; that is, it must be
observed that the micrometer wire, which should be set on at
right angles to the run of the eye-piece, coincides with the
direction of the bands; if it does not, slit and bi-prism must
be readjusted.

When two images of the slit are formed by any of the
arrangements that have been described, a complete knowledge

of the apparatus used would enable us to find the position, along the bench, of the images, and their distance apart. The distance apart of the images is, however, a very small quantity, and depends, in general, on some quantity that it is not easy to measure with accuracy. Thus, in the case of the bi-prism, the obtuse angle would be required, and we cannot be sure that the faces are quite plane, so that the angle between them, far away from the obtuse edge, is the same as the angle quite close to this edge, which is the actual angle used. In the case of Fresnel's mirrors, the angle between the mirrors would be required.

In any case, it is best to determine the distance between the pair of sources independently as follows : Place a convex lens between the eye-piece and the apparatus forming the images of the slit. Using these images as our object, place the lens, it being of short enough focal length, in such a position as to form an image of them in the eye-piece, and with the eye-piece measure the distance between the two parts of this image. There will be another position of the lens, nothing else being moved, which will give images in the eye-piece ; and the distance between these is measured. The distance between the two sources is the geometric mean between the distances in the two images.

**Displacement of the Bands.**—Suppose two identical sources, A and B, to be forming a system of interference bands on a screen X X. Now,
let a plate of glass be
placed in the way of the
light from A. The bands
will not now be formed
in the same positions
as before. The central
white band was at first
equidistant from A and
B. It is now formed in



FIG. 152.

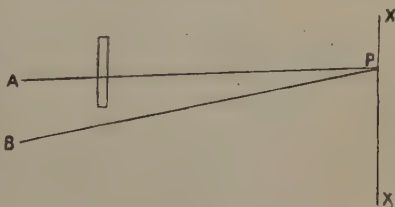such a position that its optical distances from A and B are equal, for lights of all sorts, supposing the plate to produce no dispersion. Let $e$ be the thickness of plate, $\mu$ its refrangibility, supposed to be practically the same for all light. Then, since optical distances of A P, B P are equal, we have—

$$AP - e + \mu e = BP.$$
$$(\mu - 1)e = BP - AP.$$

This is on the supposition that the dispersion produced

by the plate is negligible, so that $\mu$ is approximately the same for lights of all wave-lengths, and if P has equal optical distances from A and B for light of one colour, it will also for light of any other colour; and the various colours will be superposed at P, giving a white central band. But if the dispersion is considerable, and the thickness $e$ great, it is clear that this will not be the case. In this case there is the same number of wave-lengths in A P and B P for light of one colour; there will not be the same for light of a different colour, and the central bands for lights of various colours will not be superposed.

The plate is said to produce a *retardation* in the ray A P; for the velocity is diminished by it, and a given number of wave-lengths in the ray which passes through it corresponds to a shorter distance than in that which does not.

This displacement of the central band can only be observed in mixed light. In homogeneous light all the bands are alike, both before and after introduction of the retarding medium, and there is nothing to show which is the one corresponding to equal optical distances from the two sources.

The displacement may be used to measure the mean refrangibility of the plate. For suppose the displaced central band takes the place that was occupied by the $n$th bright band; then—

$$BP - AP = n\lambda.$$
$$\text{Thus } (\mu - 1)e = n\lambda.$$

The number of bands, with light of a given colour, by which the central band is displaced, could be found by using both white (or any mixed) light and the given light. By using white light we can measure the *distance*—with the micrometer eye-piece, say—by which the central band is displaced, supposing, as already pointed out, that the dispersion in the plate is negligible. Next, throwing the bands for the given light on the screen, we can find what number of bands occupies this distance. Then, since the central bands for white light and for the given light are displaced through equal distances, the number of bands of the given light corresponding to this distance is the number by which the central band has been displaced.

With white light only a very limited number of bands can be observed. There is a central white band, and, as we pass away to either side, a few bands with coloured edges are observed, and soon we get nearly uniform white light. For

at any point of the field, although there may be interference for light of some definite colour or colours, that is, of definite wave-lengths, yet there will be illumination received from light of all other colours.

If light is used which is nearly homogeneous, such, for instance, as would be got from a slit placed in a spectrum, a greater number of bands will be observed. But here again the light is composed of lights having a variety of wave-lengths. The field really consists of the superposition of a number of systems of bands belonging to each particular wave-length of the light used, the quality of the light varying from one edge of the slit to the other, however narrow it may be. For a considerable distance on each side of the central band, the dark bands, as also the bright bands, for all the wave-lengths coincide approximately; and thus we get a clearly marked system of bands. But, the bands for the longer wave-lengths being the broader, passing away from the centre we come in time to a part where the dark bands for some wave-lengths coincide with the bright for other, and we get uniform illumination, the bands fading out. In order to get a large number of bands, it is clear that light which is very nearly homogeneous must be used.

But there is another circumstance that may set a limit to the number of interference bands, even if quite homogeneous light were used. Suppose that the type of the vibrations of the points in the sources gradually changes, that is, the straight lines or the curves in which the ether-particles vibrate change. Then, if we take a point in the field such that the difference of its distances from the sources is so great that during the time light takes to travel a length equal to this difference the type of the vibration changes considerably, the lights on reaching this point are not in a condition to interfere, and we should in time get a field of uniform illumination.

Experiments have been made with a view to determining whether a limit can be set to the number of vibrations performed by an ether-particle before the type has changed. This would be the same as the number of interference bands, formed by homogeneous light, counting from the central one, before the field begins to be uniformly illuminated.

Fizeau and Foucault used Fresnel's two mirrors to produce bands. They placed a slit at the central white band, and analyzed the light passing through by means of a spectroscope. By continuously moving one of the mirrors parallel to itself, the system of bands passed over the slit, so that the light of

any part of the system could be examined. When the slit is at the position of the central band, the spectrum is continuous. As it moves away, it first comes to the place where the violet light first interferes, and a dark band appears in the violet end of the spectrum and passes down to the red, to be followed by others. Then two or more bands appear together in the spectrum, and they follow each other more and more quickly as the slit is moved further away from the centre. For when the slit is at a point, the difference of whose distances from the sources is an odd multiple of any half wave-length, the light of that wave-length will interfere at that point; and as we get farther away from the centre and the above difference increases, it becomes an odd multiple of more and more half wave-lengths.

Without counting all the dark bands, the number that has come into the spectrum may be determined. Suppose that in some position of the slit in the system two bands are observed in the spectrum, in places corresponding to known wave-lengths $\lambda$, $\lambda'$. Suppose these two to be the $n$th and $n'$th, $n$ and $n'$ being unknown. Now, suppose there are N bands, counting from one of these to the other, so that $N = n' - n$. And let $x$ be the unknown difference of the distances of the slit from the two sources. Then $x$ is an uneven multiple of $\frac{\lambda}{2}$ and of $\frac{\lambda'}{2}$. Thus we have—

$$x = (2n - 1)\frac{\lambda}{2}; \; x = (2n' - 1)\frac{\lambda'}{2}.$$
$$N = n' - n.$$

From these—

$$N = x\left(\frac{1}{\lambda'} - \frac{1}{\lambda}\right);$$
$$= \frac{N\lambda'\lambda}{\lambda - \lambda'}.$$

By this method it has been found that interferences are produced with a difference of path equal to 40,000 wave-lengths of violet light.

By using sodium light it is said that as many as 200,000 bands have been observed.

**Interference Refractometer** is the name given to an apparatus by means of which indices of refraction may be measured by finding the displacement produced in interference bands, according to the principle just described. The chief

object to be attained in such an apparatus is to have two interfering beams of light coming from identical sources, or originally from the same source, these beams being considerably separated from each other throughout a portion of their paths, so that a retarding medium may be interposed in the way of one without affecting the other.

In M. Jamin's apparatus two thick glass plates P B, C D, of as nearly as possible equal thickness, and silvered at the back, are carried in vertical positions on supports resting
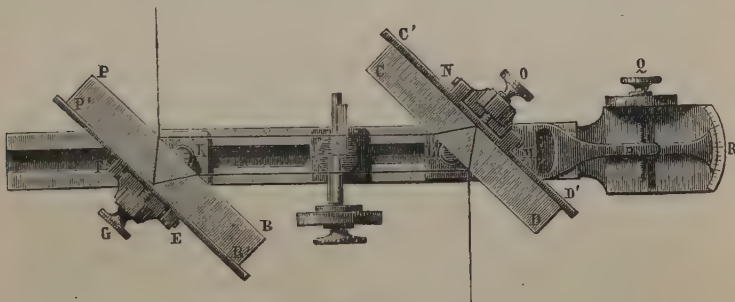


FIG. 153.

in the same groove. The plates are parallel to each other and inclined at $45°$ to the groove. P B is fixed, and, by means of the screw O, C D can be set accurately vertical, and by means of Q it can be turned about a vertical axis.

Now, suppose the plates to be in exact adjustment, and imagine a narrow pencil of light falling on the first. This will be partly reflected from the front and partly from the back surface, and these parts, after reflexion from the back and front surfaces respectively of the second plate, will have traversed equal optical paths, and thus emerge in the same phase. Now, suppose that by the screw O the plate C D is set slightly out of adjustment, so that the plates now have their surfaces intersecting in horizontal lines. The emergent rays will, as a rule, have traversed different optical distances, and will for some differences be in complete discordance, producing interference. Suppose, then, that P B is illuminated by light from a large source, as from the sky. Then if the eye looks at C D, rays will be received in all directions coming from C D, and in certain of these directions there will be interference and in others reinforcement. A series of horizontal interference bands will then be seen. And by adjusting the screw

O so as to set C D more or less parallel to P B, these bands will be made wider or narrower. By moving the screw Q so as to turn C D about the vertical, the central band will be displaced, and the entire system of bands shifted up or down. In the same way, if one of the rays between the two plates is retarded relatively to the other, the same effect is produced.

In order to measure the retardation produced in the ray in the paths of which a retarding medium is placed, an apparatus called a *compensator* may be employed. Such an apparatus is used to produce relative retardation of the rays of the opposite sense and of the same amount as that produced by the medium used, so that the displaced bands are brought back to the old positions; and then the adjustment of the compensator gives the retardation produced.

One form of compensator consists of two plates of glass through which the two rays pass normally. If these plates are of the same thickness, they introduce no relative retardation. One of these two plates is made of two wedges slightly tapering in opposite directions, so that, by sliding one over the other, a plate of any desired thickness may be obtained, and thus the required relative retardation produced.

Jamin's compensator, which is represented as being used with his refractometer, consists of two glass plates fixed to a common axis in the manner shown in Fig. 153, and with a small angle between them. This arrangement is set so that the two rays pass through the two plates. The retardation produced by a plate will increase with the obliquity at which it is met by the ray. It is found that when this compensator is turned about its axis, the displacement it produces in the bands is almost proportional to the angle by which it is turned. By altering the angle between the plates, the sensitiveness of the compensator can be altered.

The relative retardation produced by a compensator set in any manner could be calculated from a knowledge of the dimensions and refractive index of its parts. But it is better to graduate it by experiment, observing the displacements produced in the bands for various settings of the compensator.

By means of this apparatus, Jamin has compared the refractive indices of various gases with that of air, and those of dry and of moist air.

### EXAMPLE.

Fresnel's fringes are produced with homogeneous light of wave-length $6 \times 10^{-5}$ cms. A thin film of glass (refractive index $1 \cdot 5$) is introduced

into the path of one of the interfering rays, upon which the central bright band shifts to the position previously occupied by the fifth bright band from the centre (not counting the central band itself). The ray traverses the film perpendicularly. What is the thickness of the glass film? (Science and Art Hons., 1894.)

# CHAPTER XIII.

## *DIFFRACTION.*

THE fact that light travels in straight lines was the great difficulty in the way of the wave theory. Sound, which was known to consist of waves in the air, is not cut off by an obstacle in the line between the source and the observer, but makes its way round the edges of the obstacle. Why, then, it was argued by Newton and other supporters of the emission theory, does not light behave in a similar way? The answer to this is twofold. In the first place, sound does behave in the manner which at first sight appears to be peculiar to light; that is, it may be cut off by an obstacle, but the obstacle must be of very large dimensions. A mountain range will almost entirely prevent a sound from being heard at a comparatively short distance from the source, on the other side of the range, the sound being distinctly heard at a much greater distance at points in an unobstructed line with the source. And, next, by special arrangements, it can be shown that light behaves like sound; that is, it does, to some very small extent, get round the edges of obstacles; and a source will illuminate points on a screen so situated that the straight lines joining them to the source are cut by the obstacle. The difference between the two cases is one not of kind, but of degree. The cause of the difference is to be found in the enormous difference in the wave-lengths of the undulations, the sound-waves being frequently several feet in length, and the mean lengths for light being about $\frac{1}{50000}$ inch. It will be remembered that in explaining rectilinear propagation of light, the extreme minuteness of the wave-length was found to be necessary.

We have now to deal with a class of optical phenomena which depend on the property possessed by light of deviating from the rectilinear path at the edge of the obstacle. These are called **diffraction phenomena.**

In discussing diffraction effects it will be convenient to

speak of the *geometrical shadow* of an obstacle. This means the shadow that would be formed if the paths of the rays were always truly rectilinear, and the boundary of which would be formed by drawing straight lines from the source to touch the edge of the obstacle.

As the first of these phenomena, we shall consider the following: Imagine monochromatic light to proceed from
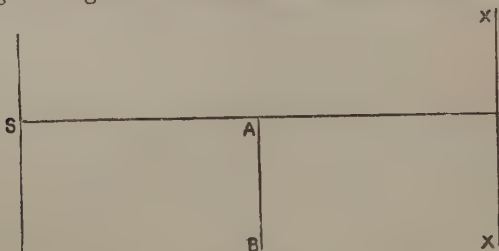


Fig. 154.

a very narrow slit denoted at S, and to encounter an obstacle, A B, with a straight edge at A, placed parallel to the slit; a screen, X X, being placed at some distance further on, or else the light on passing by the edge being received in an eye-piece. It is found that, instead of having a shadow of A B with a definite boundary-line between the obscure and illuminated parts of the screen, there is formed a series of dark lines near the edge of the shadow, which get closer and narrower as we pass away from the edge, and gradually fade away into uniform illumination.

These dark lines were observed by Young, and he attempted to explain them by supposing that they were due to the interference of the waves coming direct from the source with those which are reflected from the edge of the obstacle. Fresnel showed that this was not the true explanation, for he found that the lines are formed in just the same position whether the edge is sharp or rounded. And it is to Fresnel that we owe the true theory of diffraction phenomena. The basis of the theory is this: Suppose that each little element of any wave-front of light proceeding from a given source acts, when the vibration has reached it, as a source of vibration, thus sending out *secondary waves* into space. If, when the wave-front has reached a given surface, it finds there an obstacle, certain of these secondary sources are destroyed; and the diffraction effects are produced by mutual interference of the light from the other sources.

Let us consider how the bands produced by an edge may be explained by this theory.

Take a point, P, on the geometrical shadow of the obstacle. The wave-front through A of the vibrations from S gives, as traced on the paper, the circle A C. We must suppose that all
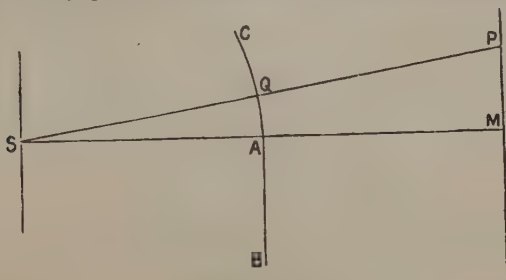


FIG. 155.

the points on this wave-front become, when the vibrations from S reach them, centres of vibrations, which they send on to the screen. Now, it is only a small portion of the wave-front, close round Q, that produces any effect at P. Thus, unless the line S P falls very near to A, the illumination at P will be the same as if the obstacle were removed. If, however, Q lies very near to A, the screen cuts off some portion of the wave-front that would produce an effect at P.

The various points along Q A are in the same phase of vibration at any instant, and they send these vibrations to P in times depending on their distances from P. Thus the displacement at P at any instant is the resultant of an indefinite number of infinitesimal displacements of different phases; and to give an exact account of the phenomenon, it must be considered in this way. However, this and other cases in diffraction may be more simply, although less exactly, explained by the following method :—

Along the arc Q A take points $R_1$, $R_2$, etc., so that—

$$PR_1 - PQ = PR_2 - PR_1 = \ldots = \frac{\lambda}{2}$$

Then we may regard each one of the elements $Q R_1$, $R_1 R_2$, etc., as approximately consisting of points equidistant from P; and the difference of the distances from P of two consecutive elements is $\frac{\lambda}{2}$. Thus two consecutive elements send vibrations to P which are just in opposite phases; and the elements

tend to destroy each other's effect at P in pairs. These parts into which the wave-front is thus cut off are called *half-period elements* with respect to P, the vibrations they send to P reaching it at times differing by half a period. If there is one half-period element in Q A, P is brighter than if just in the continuation of S A. If there are two, the second nearly destroys the effect of the first, and P is a point of maximum darkness. If there are three, P becomes a bright point again; and so on. Thus the condition for maximum brightness at P is—

$$AP - QP = (2n + 1)\frac{\lambda}{2}.$$

And the condition for minimum brightness is—

$$AP - QP = 2n \cdot \frac{\lambda}{2}.$$

As the screen is moved about, the points in which any given bright or dark band meets it, in a given plane at right angles to the slit, lie on a hyperbola. For, considering a dark band—

$$AP - QP = n\lambda.$$
$$\text{Thus } SP - AP = SA - n\lambda.$$

That is, P lies on a hyperbola with S and A as foci.

If P is within the geometrical shadow, divide up the arc A Q into half-period elements, starting from A. If P is near to M, the boundary of the geometrical shadow, the first element has a much greater effect at P than any other, for it is much larger than any other, besides being less oblique to A P. As P moves farther into the geometrical shadow, the effects at P of the elements become more and more nearly equal, so that they more and more nearly cut each other out in pairs. Thus the brightness falls off uniformly from the edge of the geometrical shadow, passing inwards, without any alternations.

As in the case of interference, to obtain bands it is clear that we must have a source of very small dimensions, such as a small aperture or a narrow slit parallel to the edge A; for if the dimensions of the source are considerable, the various points of it will give rise to separate systems of fringes, which will be so superposed as to obliterate each other.

To calculate the positions of the bands outside the geometrical shadow of the edge, put SA = *a*, AM = *b*; and

suppose the wave-length of the light used is λ. Then we have—

$$SP - AP = SA - m \cdot \frac{\lambda}{2}$$

where $m$ is odd or even for a bright or a dark band respectively.

Let $x$ denote the distance, P M, of the corresponding band from M. Then—

$$AP^2 = b^2 + x^2 ;$$
$$AP = b \left( 1 + \frac{x^2}{b^2} \right)^{\frac{1}{2}}$$
$$= b + \frac{x^2}{2b}, \text{ approximately.}$$

And—

$$SP = a + b + \frac{x^2}{2(a + b)}, \text{ approximately ;}$$
$$\therefore QP = b + \frac{x^2}{2(a + b)}.$$

Thus since $AP - QP = m\frac{\lambda}{2}$, —

$$\frac{x^2}{2b} - \frac{x^2}{2(a + b)} = m\frac{\lambda}{2} ;$$
$$\therefore x = \sqrt{\frac{b(a + b)}{a} m\lambda.}$$

For bright bands—

$$x = \sqrt{\frac{b(a + b)}{a}(2n + 1)\lambda.}$$

For dark bands—

$$x = \sqrt{\frac{b(a + b)}{a} 2n\lambda.}$$

If, instead of using monochromatic light, white light is used, we shall get systems of bands arising from the light of each colour; and these will be superposed. The breadths of the bands will increase with the wave-length. Thus, in the resulting effect, we shall see a few bands and coloured edges. The first bright band for red light being wider than for any other colour, the edge of the resulting band turned away from the shadow will appear red. Passing across the first dark band, we come to illumination by the violet light first. Thus the edge of the next bright band turned towards the shadow

will be coloured blue. The same applies to the other bands. The colouring of the bright bands will thus be red away from the shadow, and blue towards the shadow. Only a few bands will be seen. They very soon fade away into nearly uniform illumination.

The diffraction bands produced by an edge may be conveniently observed by means of the optical bench. To verify the laws of them, the following arrangement is suitable: A
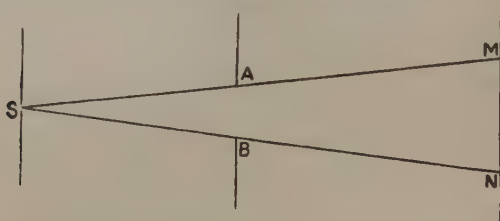


FIG. 156.

narrow slit, S, at one end of the bench is illuminated by means of monochromatic light. At A B is an aperture with edges parallel to S, and pretty broad, so that one edge may not interfere with the bands formed by the other. Suppose M N is the focal plane of the eye-piece. Then, by knowing A B, and the distances of S from A B and from M N, we can determine the breadth of the geometrical image of A B. To find the distance of any band from the geometrical shadow of the corresponding edge, calculate the distance between the two geometrical shadows and observe the distance between two corresponding bands. Half the difference between these two quantities is the distance from a band to the shadow of the edge. In this way the positions of all the bands that can be observed may be determined with reference to the edge.

Suppose monochromatic light from a narrow slit, S (Fig. 157), passes through a narrow aperture, A B, with edges parallel to the slit, and falls on a screen, X X, parallel to the plane of the aperture. Let S M, perpendicular to aperture and screen, pass midway between A and B. To determine the effect at a point P, outside the geometrical image of the aperture, divide the wavefront at A B into half-period elements with respect to P. P will be at the centre of a bright or a dark band, according as the number of half-period elements is odd or even; that is, according as PB − PA is an odd or an even multiple of $\frac{\lambda}{2}$.

Now, suppose the breadth A B of the aperture to be $b$, and let its distance from the screen be $a$. Let MP = $x$. And
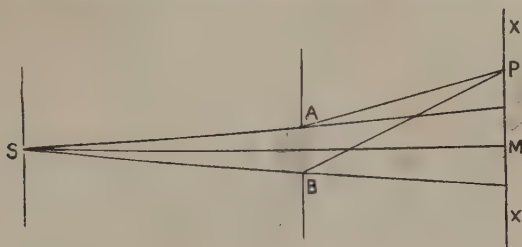


FIG. 157.

suppose P is the centre of a dark band, so that PB — PA = $n\lambda$. Then, just as in the case of interference, we have—

$$x = \frac{n\lambda a}{b}.$$

This indicates a series of dark bands equally spaced out from each other. The bright bands between these are given by—

$$x = \frac{(2n + 1)\lambda a}{2b}.$$

These bands are very easily produced without special apparatus. They may be seen by looking edgewise at a gas-flame several feet off through a narrow slit in a piece of paper. Several coloured images will be seen to right and left of the flame; and, as the slit is narrowed, they broaden out and separate from each other.

They may also be produced on the optical bench by using a slit as source, and having between it and the eye-piece another slit that can be made very narrow. This slit must now be made much narrower than for the experiment already described, in which each of its edges formed bands separately. As the slit is narrowed up, we pass from that case to the present one. The difference in the appearance of the bands in the two cases is very marked. In the first, all the bands are spaced out in a field of uniform illumination, and the distances between them are unequal. In the second, the narrow space just opposite to the slit is by far the brightest; the rest of the field is much darker than in the other case, and the bands are equidistant.

Suppose, instead of a screen with an aperture, we have

a small obstacle A B, such as a wire, with sides parallel to the slit.

Outside the geometrical shadow, on each side will be observed a system of diffraction bands, the same as those
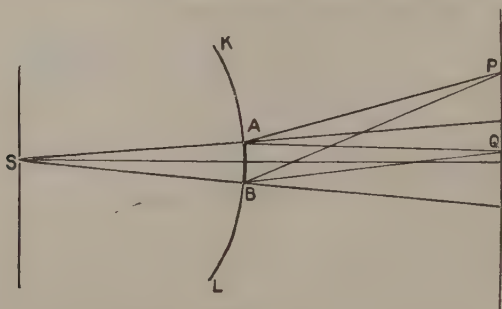


FIG. 158.

produced by a straight edge. That on the side of A, at such points as P, is produced by the light that passes by the edge A, acting independently of the edge B; B P being too oblique to the wave-front B L for this to produce any appreciable effect at P as compared with the effect produced by A K.

Inside the geometrical shadow another system of bands will be observed. These, unlike the others, are at equal distances apart. A point, Q, receives light from both wavefronts A K and B L, the effective portion of either being practically limited to a few half-period elements at the edge. The action at Q will then be the same as that of two identical sources at A and B. In fact, we may suppose Q to be receiving light along two paths, S A Q, S B Q. There will then be interference bands formed within the shadow; bright bands being given by—

$$QB - QA = n\lambda,$$

and dark bands by—

$$QB - QA = (2n + 1)\frac{\lambda}{2}.$$

If $b$ is the breadth of the obstacle, and $a$ its distance from the screen, the distance between two consecutive bands, either bright or dark, will be $\frac{a\lambda}{b}$.

The bands produced in this case, both within and without the geometrical shadow, are easily observed on the optical

bench. The breadth of the shadow may be calculated as in the case of an aperture, and the distances of the bands from its edges determined as for that case. To observe the bands inside the shadow well, a very strong light is necessary.

We shall now investigate, in a more exact manner, the effect produced by a narrow aperture with sides parallel to the
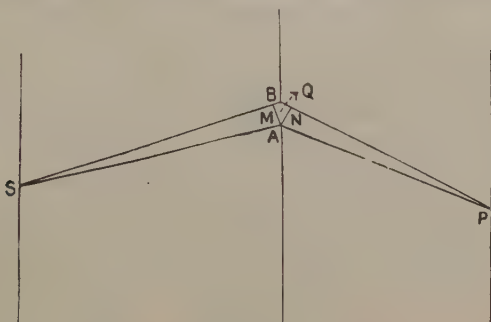


FIG. 159.

slit through which the light comes. The plane of the paper is at right angles to slit, aperture, and screen. Let A B be the aperture; S, the slit; P, any point on the screen.

Suppose S and the screen both so far from the aperture that we may suppose S A and S B parallel, and also P A and P B. Suppose the lines S A, S B make an angle, $i$, with the normal to the plane of the aperture; and the lines P A, P B an angle, $\theta$, with the same normal. Let $AB = d$, let the distance $SAP = l$. From A draw A M, A N perpendicular to S B, B P. Then $MB = d \sin i$; $BN = d \sin \theta$.

$$\therefore SBP = l + d(\sin i + \sin \theta).$$

Similarly, if Q is any point in AB such that $AQ = y$—

$$SQP = l + y(\sin i + \sin \theta).$$

To find the full effect at P, we must divide A B up into infinitesimal strips, such that the disturbances from S, reaching P through the various points of one of them, may reach P in the same phase.

Suppose S to be emitting disturbances of period T, wavelength $\lambda$, and amplitude $a$. The disturbance which reaches P in the time $t$ by the path S Q P is proportional to—

$$a \sin 2\pi \left\{ \frac{t}{T} - \frac{l + y(\sin i + \sin \theta)}{\lambda} \right\}.$$

P

Divide up $d$ into a large number, $n$, of equal strips, so that we may suppose the length of path in the plane of the paper, from S to P through any point in one of them, to be the same.

Then the displacement at P, which has been propagated through the $r$th strip from A, is equal to—

$$k \sin 2\pi \left\{ \frac{t}{T} - \frac{l}{\lambda} - \frac{rd}{n\lambda}(\sin i + \sin \theta) \right\} \frac{d}{n},$$

where $k$ depends on the light emitted by S, and on the distances of S and screen from aperture.

The entire displacement is got by adding such quantities as this for all values of $r$ from 1 to $n$, and making $n$ indefinitely great.

Now, by a proposition in trigonometry—

$$\sin a + \sin (a + \beta) + \ldots + \sin (a + \overline{n-1}\beta)$$
$$= \frac{\sin (a + \frac{n-1}{2}\beta) \sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}}.$$

In the sum we require we have, instead of $a$, $2\pi\left(\frac{t}{T} - \frac{l}{\lambda}\right)$, and, instead of $\beta$, $-2\pi\frac{d}{\lambda n}(\sin i + \sin \theta)$.

The required sum is thus—

$$\frac{k \sin 2\pi \left\{ \frac{t}{T} - \frac{l}{\lambda} - \frac{n-1}{2} \cdot \frac{d}{n\lambda}(\sin i + \sin \theta) \right\} \sin \left\{ \frac{\pi d}{\lambda}(\sin i + \sin \theta) \right\}}{\sin \left\{ \frac{\pi d}{n\lambda}(\sin i + \sin \theta) \right\}} \cdot \frac{d}{n}.$$

This becomes in the limit—

$$\frac{k \sin 2\pi \left\{ \frac{t}{T} - \frac{l}{\lambda} - \frac{d}{2\lambda}(\sin i + \sin \theta) \right\} \cdot \sin \left\{ \frac{\pi d}{\lambda}(\sin i + \sin \theta) \right\}}{\frac{\pi d}{\lambda}(\sin i + \sin \theta)} \cdot d.$$

We see from this expression that the resulting displacement at P is in the same phase as that which would be received by way of the centre of the aperture, and the amplitude of the displacement will depend on the value of $\frac{\sin \phi}{\phi}$, where $\phi$ is written for $(\sin i + \sin \theta) \cdot \frac{\pi d}{\lambda}$.

The intensity of illumination at P is proportional to $\frac{\sin^2 \phi}{\phi^2}$. We must then consider the changes in the value of this expression. The intensity is zero when $\sin \phi = 0$, that is, $\sin \phi = m\pi$; or $\sin i + \sin \theta = \frac{m\lambda}{d}$. In this, $m$ has any integral value except 0. When $m = 0$, $\frac{\sin \phi}{\phi}$ has its greatest value, and is 1.

The intensity is a maximum, by elementary differential calculus, when $\tan \phi = \phi$. The corresponding values of $\phi$ have been calculated by Schwerd, and are given as follows :—

$$\phi_0 = 0 \qquad\qquad \frac{\phi_3}{\pi} = 3 \cdot 4709$$
$$\frac{\phi_1}{\pi} = 1 \cdot 4303 \qquad \frac{\phi_4}{\pi} = 4 \cdot 4774$$
$$\frac{\phi_2}{\pi} = 2 \cdot 4590$$

The values of $\phi$, after $\phi_0$, are nearly the successive odd multiples of $\frac{\pi}{2}$, and become more and more nearly equal to them. The sine of any $\phi$ is thus nearly 1 ; and the intensities are approximately proportional to the quantities—

$$1 \cdot \left(\frac{2}{3\pi}\right)^2 ; \ \left(\frac{2}{5\pi}\right)^2, \text{ etc.}$$

Thus the first maximum, for $\sin i + \sin \theta = 0$, just opposite to the slit, is by far the greatest, and the others rapidly fall off.

Suppose, next, that instead of one, we have two slits with parallel edges, each of breadth $d$, and with an opaque interval of breadth $b$ between them ; the source, slits, and screen being still so situated that we may suppose rays from a point of the source to all points in the slits lying in a plane at right angles to them to be parallel, and rays from these points to a point on the screen to be parallel.

Thus A B, C D are the apertures, $AB = CD = d$; and $BC = b$.

Then, as above, $SCP = l + (d + b)(\sin i + \sin \theta)$. And the resulting disturbance at P from C D is—

$$\frac{k \sin 2\pi \left\{ \frac{t}{T} - \frac{l}{\lambda} - \frac{1}{\lambda}\left(d + b + \frac{d}{2}\right)(\sin i + \sin \theta) \right\} \cdot \sin \left\{ \frac{\pi d}{\lambda}(\sin i + \sin \theta) \right\}}{\frac{\pi d}{\lambda}(\sin i + \sin \theta)} \cdot d,$$

The entire disturbance at P is got by adding the disturbances for A B and C D; and is, putting $\phi$ for $(\sin i + \sin \theta) \cdot \dfrac{\pi d}{\lambda}$,—
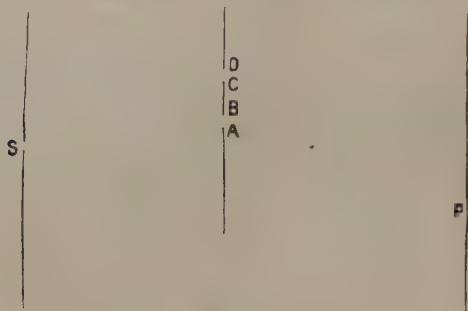
$$2k\sin 2\pi\left\{\frac{t}{T}-\frac{l}{\lambda}-\frac{2d+b}{2\lambda}(\sin i+\sin\theta)\right\}\cos\frac{\pi(d+b)(\sin i+\sin\theta)}{\lambda}\cdot\frac{\sin\phi}{\phi}\cdot d$$



FIG. 160.

Thus the disturbance at P has the same phase as if all the waves came by the path S O P, where O is the middle point of B C; for $\text{SOP} = l + \left(d + \dfrac{b}{2}\right)(\sin i + \sin \theta)$. And the intensity is proportional to the square of—

$$\cos\frac{\pi(d+b)(\sin i+\sin\theta)}{\lambda}\cdot\frac{\sin\phi}{\phi}.$$

There will then be other points of zero illumination, given by—

$$\cos\frac{\pi(d+b)(\sin i+\sin\theta)}{\lambda}=0,$$

or—

$$(d+b)(\sin i+\sin\theta)=(2m+1)\frac{\lambda}{2}.$$

Thus there is, superposed on the dark bands given by $\sin \phi = 0$, a system of dark bands depending on the distance apart, $d + b$, of the slits. It is easy to see how these bands are formed: it is by the interference of the lights coming through the two slits. For if K, L are two corresponding points in A B, C D; so that AK = CL; then KL = $d + b$, and with the condition just stated between $\theta$, $i$, etc., the paths S L P, S K P differ by an odd number of half wave-lengths; and so for paths through any two corresponding points of A B, C D.

**Grating.**—We have next to consider the effect at P when there is a large number of apertures, equal in width and at equal distances, on which the light from the slit falls. This is the case of the **optical grating**. As before, let the breadth of each aperture be $d$, and that of the opaque space between two, $b$. If we take the sum of the displacements due to $p$ such apertures, we shall see that the resulting disturbance at P is in the same phase as that which comes by way of the middle point of the grating. The resulting displacement will further contain the factor—

$$\frac{\sin \dfrac{p\pi(d+b)(\sin i + \sin \theta)}{\lambda}}{\sin \dfrac{\pi(d+b)(\sin i + \sin \theta)}{\lambda}}.$$

This factor is a maximum, for variations of $\theta$, when $\dfrac{\pi(d+b)(\sin i + \sin \theta)}{\lambda}$ is a multiple of $\pi$; $= n\pi$, say. Then the value of the factor is $p$. Thus by putting $(d+b)(\sin i + \sin \theta) = n\lambda$, we find points of maximum intensity on the screen.

Further, the illumination at these points is very much greater than anywhere else. For suppose $\theta$ to increase very slightly from the value giving a maximum of illumination. Suppose $\dfrac{\pi(d+d)(\sin i + \sin \theta)}{\lambda}$ to become $n\pi + x$. Then the above factor is, numerically,—

$$\frac{\sin px}{\sin x}.$$

And $p$ being very large, the angle $px$ is considerable ; so that the value of the fraction is much less than $p$, its maximum value, even for small values of $x$. Thus the slit will give, as a rule, definite lines of great brightness on the screen.

This important case we shall also consider independently, and by means of a less mathematical treatment.

Let A A', B B', etc., be the successive apertures of the grating. The rays through corresponding points of the apertures will reinforce each other, if the lengths of the paths through these points increase or diminish, as we pass from one to the next, by $\lambda$ or any multiple of $\lambda$. From B, C, etc., draw B K, C L, . . . and B M, C N, . . . perpendicular to the two parts of the ray through A. Then the rays shown in the figure will reinforce each other if KAM $= \lambda$, LAN $= 2\lambda$, etc. ; or, generally if KAM $= n\lambda$, LAN $= 2n\lambda$, etc. In

this case the rays through any corresponding points of the apertures, and parallel to those drawn, will also reinforce each
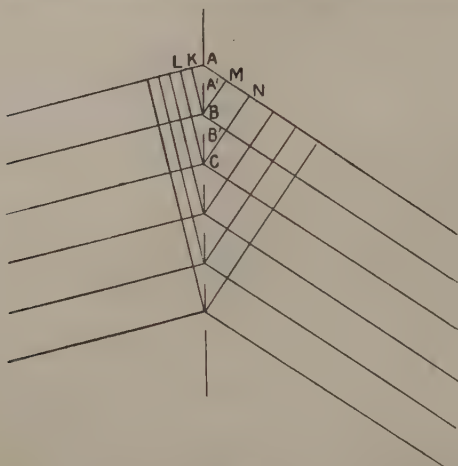


FIG. 161.

other in the given direction. Thus if a single pair of apertures will produce illumination at a given point of the screen, a grating will produce much more.

In the figure $\angle$ KBA $= i$; $\angle$ ABM $= \theta$.

$$\text{Thus KAM} = (d + b)(\sin i + \sin \theta);$$
$$\text{LAN} = 2(d + b)(\sin + i + \sin \theta); \text{ and so on.}$$

Thus—if KAM $= n\lambda$,

$$n\lambda = (d + b)(\sin i + \sin \theta).$$

Or, if there are N lines to the grating per unit of length, $d + b = \dfrac{1}{N}$, therefore—

$$\sin i + \sin \theta = Nn\lambda,$$

where $n$ is a whole number.

If light is incident normally on the grating, we have—

$$\sin \theta = Nn\lambda.$$

It may happen that for a certain value of $\theta$ given by the above formula, the width of an aperture is such that it produces no illumination in that direction. This will be the case if the rays from the various points of the aperture cut each other out in pairs, that is, as we have seen already, if $d$ $(\sin i + \sin \theta)$

is a multiple of $\lambda$; say $= m\lambda$. If the apertures are all of the breadth $d$ satisfying this condition—

$$d(\sin i + \sin \theta) = m\lambda,$$

and for the same value of $\theta$ we have—

$$(d + b)(\sin i + \sin \theta) = n\lambda,$$

then the $n$th bright line will be absent.

It is easy to see how distinct images of the slit are formed in definite positions, without any gradual fading away of the intensity on either side. For suppose we have a point of maximum intensity for a certain value of $\theta$ determined as above. Then the disturbances reaching this from the various apertures are all in the same phase. Now, suppose we take another point on one side but quite close to this. The disturbances reaching this point through two consecutive apertures are not quite in agreement. And if we choose two apertures, the difference of whose distances from the point is $\frac{\lambda}{2}$, the disturbances reaching the point through them are in complete disagreement, and neutralize each other. Call these two apertures the first and the $(r + 1)$th. Then, in the same way, the second and the $(r + 2)$th will neutralize each other's effect at the point, and so on. The point will only receive illumination from the apertures that are left over after picking out those that neutralize each other. If, then, there is a very large number of apertures, they will mostly destroy each other's effect at any point except those lying in the positions of maximum illumination; and in these positions we get intensity, compared with which that at the other parts of the screen is negligible.

We have been supposing that slit and screen are both so far from the grating that we may suppose all the rays from the slit, or to a given point of the screen, to be parallel. Parallelism is attained in practice without using very great distances, by the help of lenses. A convex lens is placed at L, so as to have its principal focus at S, and to direct the light to the grating; and another is placed at L′ in the path of the light from the grating, and so that P is its principal focus. Then the action will be the same as that already described. For the optical lengths of the paths from S to all points in the wave-front between L and the grating will be equal. And this wave-front is perpendicular to the rays from L to the grating. Thus the optical differences of the

paths from S to corresponding points in consecutive apertures is, as before, $(d + b) \sin i$. In the same way, the optical
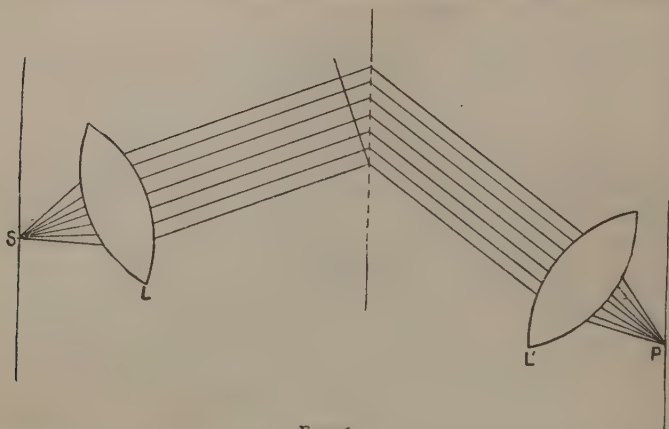


FIG. 162.

differences of the paths from these points to P is $(d + b) \sin \theta$. Thus we get maximum illumination at P if $(d + b)(\sin i + \sin \theta)$ is a multiple of $\lambda$.

These grating images may be seen without any very special apparatus. If a small source of light be looked at through a fine silk handkerchief, a few images will be seen on each side of the source. They may be seen on looking at a gas-light through the silk cover of an umbrella. Very fine wire gauze will also produce them. In these cases the retina of the eye is the screen on which the images are thrown, and the lens of the eye takes the place of the lens L'.

Suppose white light to be sent through the slit to the grating, with the arrangement of lenses just explained. Then each separate colour will give rise to a series of diffraction images. In any one set of images of the various colours the separate coloured images will be arranged side by side, the positions of the images varying continuously with the wave-lengths, those of longer wave-lengths being less deviated than those of shorter. Thus we shall get a series of coloured spectra. In any one spectrum the position of any colour will depend on the wave-length of the colour, the number of lines per unit length of the grating, and the arrangement of the apparatus. It will not depend upon the nature of any material used, as the relative positions of the colours in a

prismatic spectrum depend upon the nature of the glass. ·If the incident light falls normally on the grating, we have the simple law, that in any spectrum the sine of the angle of deviation of any colour from the direct image of the slit is simply proportional to the wave-length of that colour. For this reason, grating spectra are frequently used as standards of reference rather than prismatic spectra.

The optical grating affords the most accurate method of measuring the wave-length of light of a given quality. For this purpose it is used with a spectrometer. The slit of the collimator is illuminated with light of the given quality. The collimator and telescope of the spectrometer are set for parallel light, and the grating is set on the table of the spectrometer with its lines normal to the table. To do this, first get the face normal. This may be tested, as in setting the prism on the spectrometer, by observing whether the reflected image of the slit is formed at the proper level. Next, if necessary, turn the grating in its own plane, to get the lines right. This adjustment is complete when the diffraction images are all formed at the same level. The measurements may now be made in either of two ways.

1. Set the grating normal to the incident light. Set the telescope to view any image, say the $n$th, to right and to left of the direct image. Half the difference between the readings of the telescope is the angular deviation of the image. Call this deviation $\theta_n$.

The number of lines of the grating per unit of length (say per millimetre) is measured. This may be done by means of a microscope, using the micrometer eye-piece and a standard micrometer scale. Let this number be N. Then we have, if $\lambda$ is the required wave-length—

$$\sin \theta_n = n N \lambda.$$

2. If the normal to the grating makes an angle $i$ with the incident light on one side, and an angle $\theta$ with the light going to the $n$th image on the other, we have—

$$\sin i + \sin \theta = n N \lambda;$$
$$\therefore 2 \sin \frac{i + \theta}{2} \cos \frac{i - \theta}{2} = n N \lambda.$$

The angular deviation of the $n$th image from the light coming from the slit is in this case $i + \theta$. Then this deviation is least when $\cos \dfrac{i - \theta}{2}$ is greatest, that is, when $i = \theta$. If, then the grating is set to produce the $n$th image in the

position of minimum deviation, and this deviation is $\phi_n$, since $\phi_n = i + \theta$, we have—

$$2 \sin \frac{\phi_n}{2} = n \mathrm{N} \lambda.$$

The angle $\phi_n$ may be determined, as before, by observing the $n$th image to right and to left of the direct image.

**Reflexion Grating.**—Suppose a plane reflecting metallic surface to be ruled with a large number of fine equidistant parallel lines. This will produce diffraction images similar to those of the transparent grating. Waves falling on the reflecting grating will travel out in all directions on the side of the incident light, and in certain directions will produce very
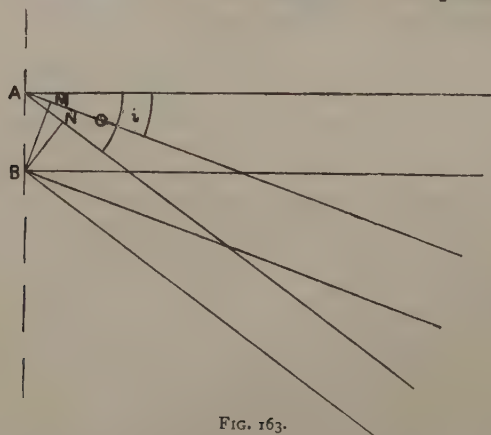


Fig. 163.

strong effects. These directions may be found in just the same manner as that already employed. Suppose there are N lines per unit length of the grating. Let the incident light make an angle, $i$, with the normal. Suppose that a maximum of effect is produced by the irregularly reflected light in a direction making an angle, $\theta$, with the normal on the same side. Let A and B be corresponding points on two of the reflecting lines. Draw B M, B N perpendicular to the incident and irregularly reflected light at A. Then a maximum of effect is produced in the direction $\theta$, by the united action of corresponding points of the various reflecting strips, when $\mathrm{MA} + \mathrm{AN}$ is a multiple of the wave-length, that is, since $\mathrm{AB} = \frac{1}{\mathrm{N}}$, when—

$$\frac{\text{I}}{\text{N}}(\sin i + \sin \theta) = n\lambda,$$

or—

$$\sin i + \sin \theta = n\text{N}\lambda.$$

If the angle $\theta$ is measured on the other side of the normal, the condition is—

$$\sin i - \sin \theta = n\text{N}\lambda.$$

The greatest effect is produced when all the points of all the reflecting strips act together, that is, in the direction of regular reflexion.

Curved gratings have been used by Professor Rowland, and by Professor Langley in his researches on solar radiation. Imagine a concave reflecting surface ruled with lines; and let the plane of the paper represent a plane cutting the reflecting strips normally at A, B, C. Suppose S is a source of light, and P a point such that the paths S A P, S B P, S C P, . . . differ by a constant multiple of a wave-length. Then an image of S will be formed at P. This image can be thrown on a screen or photographed without the assistance of a lens. If
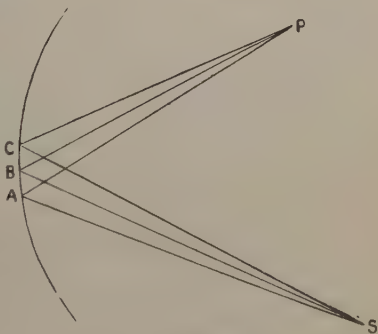


Fig. 164.

light of mixed quality be used, there will be a series of images such as P formed by the lights of various colours. Thus a spectrum is formed by means of a narrow slit placed at S. By directing various parts of the solar spectrum so formed into suitable apparatus, and especially of that part of it which is below the red, and is formed of rays whose waves are too long to produce the effect of light, Professor Langley has made his celebrated researches on this, the "infra-red" solar spectrum.

Some substances of striated structure, such as mother-of-pearl, show beautiful colours on account of acting on the incident light as reflexion gratings.

**Circular Aperture.**—Suppose light comes from a small source, S, and a screen with a circular aperture of small radius is placed so that S is on the axis of the circle. Let the radius

of the circle be $r$. Let the distance of S from the circle be $a$. To consider the illumination at a point on the axis of the
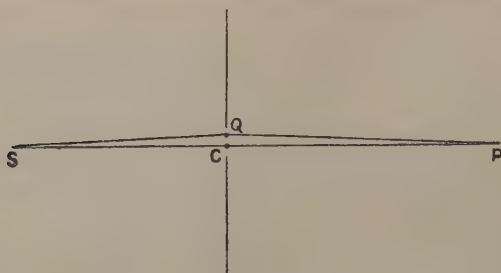


FIG. 165.

circle, at a distance $b$ from the centre, and on the side opposite to S.

Take C, the centre of the circle. Let Q be any point of it at distance $x$ from C.

$$\text{Then SQ}^2 = a^2 + x^2 ;$$
$$\text{SQ} = a + \frac{x^2}{2a} ;$$
$$\text{QP} = b + \frac{x^2}{2b} ;$$
$$\therefore \text{SQP} = a + b + \frac{x^2}{2}\left(\frac{1}{a} + \frac{1}{b}\right).$$

Suppose the area of the circle divided up into a large number, $n$, of zones equal in area, and each having centre at C. The area of each is $\frac{\pi r^2}{n}$. Each zone being very narrow, we may suppose that the vibrations from O passing through all points of it, reach P in the same phase. The displacement at P produced by disturbances reaching it through the zone of mean radius $x$, may be written—

$$k \sin 2\pi\left\{\frac{t}{T} - \frac{a+b}{\lambda} - \frac{x^2}{2\lambda}\left(\frac{1}{a} + \frac{1}{b}\right)\right\} \cdot \frac{\pi r^2}{n}.$$

If this is the $m$th zone from the centre, $\pi x^2 = m \cdot \frac{\pi r^2}{n}$; thus we write the above displacement—

$$k \sin 2\pi\left\{\frac{t}{T} - \frac{a+b}{\lambda} - \frac{mr^2}{2n\lambda}\left(\frac{1}{a} + \frac{1}{b}\right)\right\} \cdot \frac{\pi r^2}{n}.$$

Summing all such terms for all values of $m$ from 1 to $n$, and making $n$ indefinitely large in the limit, we get for the displacement at P, as in the case of the rectilinear slit—

$$k \cdot \frac{\sin 2\pi\left\{\frac{t}{T} - \frac{a+b}{\lambda} - \frac{r^2}{2\lambda}\left(\frac{1}{a} + \frac{1}{b}\right)\right\}\sin \frac{\pi r^2}{2\lambda}\left(\frac{1}{a} + \frac{1}{b}\right)}{\frac{\pi r^2}{2\lambda}\left(\frac{1}{a} + \frac{1}{b}\right)} \cdot \pi r^2$$

Putting $\phi$ for $\frac{\pi r^2}{2\lambda}\left(\frac{1}{a} + \frac{1}{b}\right)$, this becomes—

$$k \cdot \sin 2\pi\left\{\frac{t}{T} - \frac{a+b}{\lambda} - \frac{r^2}{2\lambda}\left(\frac{1}{a} + \frac{1}{b}\right)\right\}\frac{\sin \phi}{\phi} \cdot \pi r^2.$$

Thus the intensity of illumination at the distance $b$ along the axis is proportional to—

$$\pi^2 r^4 \cdot \frac{\sin^2 \phi}{\phi^2}.$$

This expression vanishes whenever $\phi = 0$, that is, when—

$$\frac{\pi r^2}{2\lambda}\left(\frac{1}{a} + \frac{1}{b}\right) = p\pi,$$

or—

$$r^2\left(\frac{1}{a} + \frac{1}{b}\right) = 2p\lambda.$$

The corresponding values of $b$ give the points of no illumination along the axis. The points of maximum illumination are found from the maximum values of $\frac{\sin \phi}{\phi}$ as in a former case.

If white light is used, the points of maximum illumination occur at different distances from the various colours; and a series of coloured points along the axis will be produced.

Without entering into the investigation for the illumination at points off the axis of the aperture, it is clear that it must be symmetrical all round the axis, and hence at any plane normal to the axis a series of rings will be formed, all circles, having for centre the point in which the axis meets this circle.

The case of diffraction with a small circular aperture has an important application, for in telescopes and microscopes the light is limited by a diaphragm with a circular aperture.

Suppose parallel light from a small source, S, proceeding in the direction of P. Take a plane through the point O, on S P,

at right angles to S P.  This plane is a wave-front of the light.
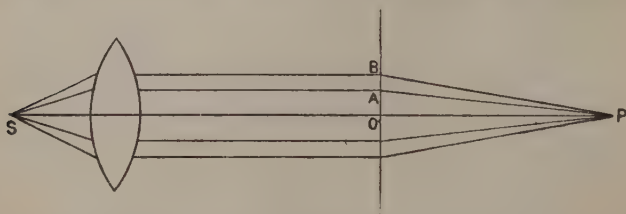Suppose this wave-front round O to be divided into half-period



FIG. 166.

elements with respect to P.  Let A, B, etc., denote the boundaries
of these elements.  Then—

$$PO + \frac{\lambda}{2} = PA = PO + \frac{OA^2}{2\,PO};$$

$$PO + \lambda = PB = PO + \frac{OB^2}{2\,PO};$$

and so on.  Thus—

$$OA^2 = 2PO \cdot \frac{\lambda}{2};$$

$$OB^2 = 2PO \cdot \lambda;$$

and so on.

Thus the areas of these half-period elements are equal,
each being $\pi\lambda \cdot PO$.  And the distances of these elements from
P O are practically equal, and their obliquities to the lines
joining points of them to P negligible.  Hence each will send
nearly the same disturbance to P; but the phases of the dis-
turbances from two consecutive ones will be opposite.

Suppose, then, that the alternate zones are rendered
opaque, so that the disturbances reaching P are all in agree-
ment.  In this way a considerable quantity of light will be
concentrated at P, as if by a lens.  Such a plate is called a
**zone-plate.**  It may be made by copying a large plate,
accurately drawn as regards relative proportions, on a much
smaller scale by means of photography.  The effect produced
by a zone-plate agrees with the theory that has just been
given of it.

**Opaque Circular Disc.**—Suppose a small opaque circular
disc, A B, to be placed before a small source, S, at right angles
to the line joining S to its centre.  To consider the illumina-
tion at a point, P, on the axis of the disc.  Take the wave-front

through A B, and divide it into half-period elements with respect to S. The first of these elements adjacent to the edge of A B produces the greatest effect at P. P, then, is illuminated



FIG. 167.
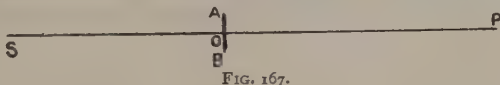
by these elements in much the same way as it is illuminated, if A B is removed, by the half-period elements into which the whole wave-front through O may be divided. And if A B is very small, the element just round its edge plays about the same part in producing disturbance at P as would be played by the first element of the full wave. Hence the illumination, in this case, at points along O P is approximately uniform, and the same as it would be if A B were removed.

M. Cornu has given a graphic method of dealing with certain cases of diffraction, which we shall now indicate.

We have seen that if a point is affected by any number of simple harmonic motions all in the same direction, the resultant is a simple harmonic motion whose amplitude and phase-angle is found by a simple construction. From any point O draw O A, A B, B C, . . . equal to the amplitudes of the S.H.M.'s, and making with any fixed direction, O X,



FIG 168.

angles equal to the phase-angles of the S.H.M.'s. Then the straight line joining O to the last point along O A B C . . . is the amplitude of the resulting S.H.M., and the angle it makes with O X is its phase-angle.
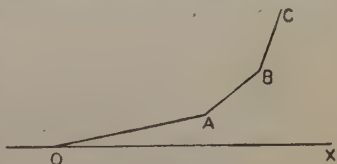
Now, imagine a small source, S, and consider the effects



FIG. 169.
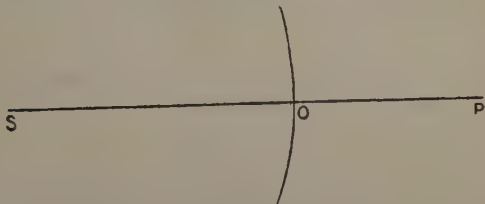
which the various parts of the wave-front from S through O produce at the point P. We shall consider only an elementary

strip of the wave-front lying in the plane of the paper, that is, between S and P. We shall thus be dealing with the effect at P of a linear wave. All the little elements of this wave are receiving disturbances from S and sending them on to P, so that the disturbances received at P from the various elements of the wave are all of the same sort, since S goes on sending out vibrations of the same sort for a long time, only they are of different phases because of the different lengths of the paths from S to P through the various elements. Now, suppose that, starting from O, we divide the wave up into equal infinitesimal elements; and suppose the phase-angle of the disturbance which the element at O sends to P is zero. The phase-angles of the disturbances from the other parts of the wave will continually increase as we pass out from O, and increase more and more rapidly as we get farther off from O. If we represent now the resulting motion at P as being due to the elementary motions from the various points of the wave, we shall get in the limit, when the elements are made indefinitely sm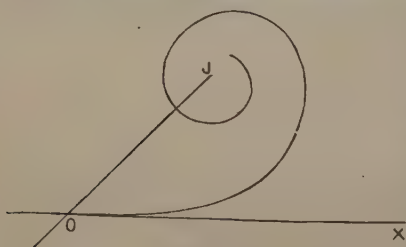all, a continuous curve, whose elements of length represent the amplitudes of the disturbances at P from the elements of the wave, and the inclinations of whose elements to a given straight line represent the phase-angles of these disturbances. Take O X as the fixed direction,



FIG. 170.

and O as the point from which to draw the curve. It starts tangentially to O X, because the phase-angle due to the element at O is zero, and it clearly curves continually in one sense, and more and more rapidly, because the phase-angle increases outward from O, and more and more rapidly. Cornu has shown that the curve is a spiral with an indefinite number of ever-narrowing turns about a point or pole, J, such that the angle XOJ = 45°.

In the same manner, the effect of the other half of the wave may be shown; and the other half of this spiral would be obtained, namely, what we should get by turning the piece O J about O through two right angles.

Now, suppose an obstacle used to intercept a portion of the linear wave. To find the effect of the remainder at various

points of a screen, with the help of the spiral, it is most con-
venient to consider the effect at a fixed point, P, of various
parts of the wave, such as would be got by moving the obstacle
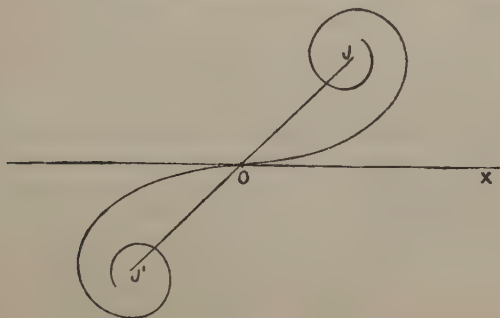about. If P occupies any given position with respect to the

obstacle, we must consider where this edge comes on the
whole wave-front in which O is between S and P. This gives
a corresponding point on the obstacle, and the disturbance at
P is represented by all the spiral on one side of this point, or
the resulting disturbance by the line joining the point to a
pole of the spiral. If there is no obstacle, the disturbance at
P is represented by J J'. If P is on the edge of the geometrical
shadow, or the obstacle comes to O, the disturbance is given
by O J. Thus the intensity is one-fourth of what it is with no
obstacle. Suppose P to move into the shadow. The dis-
turbances are denoted by the straight lines joining J to the
various points of the spiral got by moving along from O towards
J. Thus the intensity falls off continually. Suppose P to
move out from the shadow. The disturbances are denoted by
tł : straight lines joining J to the various points of the spiral
got by moving along from O towards J'. Thus the intensity
varies, passing through maxima and minima.

## CHAPTER XIV.

### COLOURS PRODUCED BY INTERFERENCE.

#### COLOURS OF THIN PLATES.

**Colours by Reflected Light.**—When light falls on a very thin film of a transparent medium with smooth reflecting surfaces, of the rays which reach the eye some will have been reflected once at the upper, or nearer, surface, and others will have been refracted into the film, reflected at the lower surface, and refracted 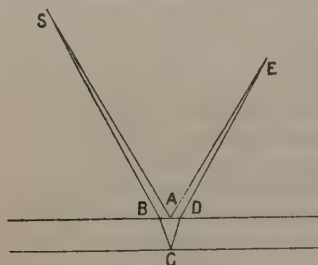out again. The light coming from a point, S, of a distant source, will thus proceed, to an eye at E, along the different paths S A E, S B C D E. And, on account of the extreme thinness of the film, these will practically come from the same point of the film; so that we may suppose the light which passes off from the film along a given straight line, and which must have reached it by parallel paths, to have all come from the same point of the source. Two such rays, then, which are reflected at the upper and under surfaces of the film, and then combine into a single ray, are in a condition to interfere; and they will interfere or reinforce each other according to the length of the difference of the optical distances of their paths. This optical distance will depend upon the nature of the film, its thickness, and the obliquity of the rays. When white light falls on the film and proceeds to the eye from a certain point of it, the optical difference of paths will generally be such as to destroy by interference the lights of certain colours only; so that the light reaching the eye will be coloured. For the light coming from another part, the obliquity and the film-thickness may both be different. And so the film will be seen of different colours in various parts. And the colour of the same part will vary according to the angle at which it is observed.

For the formation of these colours it is to be noticed that no special source of light is required; they may be observed

in daylight, or with any ordinary source of light.  They are frequently to be observed in nature, as, for instance, in mica, which has a lamellar structure.  When a drop of oil is put on the surface of still water, it spreads out into a very thin film, and shows such colours.  It is in this way that the colours of soap-bubbles are formed.  Again, when a bright piece of metal is heated, its surface shows a variety of colours, which are due to a film of un-uniform thickness of oxide of the metal that is formed on it by the action of the heat.

The colours may be produced by means of a film of air between two transparent substances.  If two thin pieces of glass, carefully freed from dust, are pressed together, they enclose between them a thin layer of air, and this will show colours ; the rays proceeding through the upper piece of glass and out into the air towards the eye, with the path-difference acquired in the film.  The pieces of glass are chosen thin merely that they may be flexible and easily pressed together.

We shall next find an expression for the optical difference of the paths of the two rays reflected at the upper and under surfaces of the film.

Let C B, F E be two parallel rays, in air, from a distant source, incident on a thin film, B D E, and leaving the film along the line B A.  Let F C be any wave-front of the light incident on the film, so that F C is at right angles to the rays, and the vibrations are in the same phase at F and C.  From E draw E K perpendicular to C B, and from B draw B M perpendicular to E D.  Then at E and K the vibrations are in the same phase.
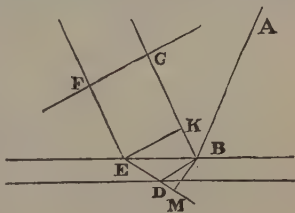


FIG. 173.

Now, let the thickness of the film be $e$, its refractive index $\mu$. And let $i$ and $r$ be the angles of incidence and refraction of the rays falling on the film.  Then the optical difference of the paths of the rays is—

$$\mu \cdot \text{EDB} - \text{KB}.$$

Now, K B in air is optically equal to E M in the film, for when the vibration at K has reached B, that at E would have reached M if the substance of the film extended to M ; or, otherwise, $\text{KB} = \mu \cdot \text{EM}$.  Thus the optical path required is—

$$\mu \cdot (\text{EDB} - \text{EM}) = \mu(\text{DB} - \text{MD})$$
$$= \mu\text{DB}(\mathbf{1} + \cos 2r)$$
$$= \frac{\mu e}{\cos r} \cdot 2 \cos^2 r$$
$$= 2\mu e \cos r.$$

It is clear from this investigation that whatever may be the medium from which the light comes to the film, the relative retardation of the rays has for its value the same expression, $2\mu e \cos r$.

This expression may also be obtained by calculating the value of $\mu \cdot \text{EDB} - \text{KB}$ in terms of $e$, $i$, and $r$.

Interference between rays of a given wave-length, $\lambda$, will then depend upon the relation which this quantity bears to $\lambda$. And it would appear at first sight that there will be interference when $2\mu e \cos r$ is of the form $(n + \frac{1}{2})\lambda$, and reinforcement when it is of the form $n\lambda$. But experiment shows that this is not the case. Because if, for example, the film becomes extremely thin, so that the expression for the difference of path vanishes, the rays, instead of conspiring, are then found to interfere, there being a minimum of illumination in this case. In the case of two pieces of glass enclosing an air-film, at a point at which the pieces are in actual contact a dark spot is observed when they are seen by reflected light. Thus the phase-difference of the two rays depends on something else besides the optical difference of their paths. This other element may be illustrated by means of a mechanical analogy. Suppose there are two perfectly elastic balls, one at rest and the other in motion, and suppose the second to collide directly on the first. It will set the first in motion and will follow it if it is heavier than the first, but will rebound if it is lighter. Now, the ether particles in two transparent media behave as if they were of different densities. And when a train of waves in one medium is falling on the boundary surface of the two, the particles at the surface, in the first, will set those in the second in motion, and will continue to move in the same direction, or will move in the opposite direction, according as the first is the denser or the rarer medium. The same thing may also be illustrated by supposing a heavy and a light chain, joined end to end, to hang vertically. Suppose the heavy chain at bottom, and let a wave be sent up it by jerking its lower end horizontally. When this wave reaches the light chain, it will be divided into two, one of which will go on into the light chain, and one will return

along the heavy one. And the returning wave will be the same as if it were sent down by jerking the upper end in the same direction as that in which the lower end was jerked. But if a wave were sent along the light chain towards the heavy one, the wave returning along the light chain one from their point of junction would be the same as if it were sent along the light chain by jerking its end in the opposite direction from that in which it was jerked to start the original wave. These two cases are called respectively reflexion of the wave *without* and *with* change of sign.

In the case of light vibrations reflected at the boundary surface of two media, they will be reflected without or with change of sign, according as the ether in the first behaves as if it were more or less dense than that in the second. If there is change of sign, the phase of the reflected vibration is exactly reversed; the effect is the same as if the waves were retarded by just half a period, or as if there were added to their path a half wave-length. To subtract half a wave length would, of course, produce just the same effect on the phase.

Now, in case of a thin film between two portions of the same medium, reflexions take place at the two surfaces of the film under opposite conditions, and there must be change of sign in one case and not in the other. The result is then the same as if the difference of the paths which we have considered is simply increased by $\frac{\lambda}{2}$. We can therefore write for the relative retardation—

$$2\mu e \cos r + \frac{\lambda}{2}.$$

We thus see that we shall get interference or reinforcement according as $2\mu e \cos r$ is of the form $n\lambda$ or $n\lambda + \frac{\lambda}{2}$.

A thin film may be situated between two different media which are such that light falling on the film is reflected at both its upper and lower surfaces under the same conditions, that is, in both cases with or in both cases without change of sign, so that no difference of phase is introduced by the reflexions. Now, it is reasonable to suppose that if light is reflected in a medium at the boundary of it and an optically denser one in which light travels more slowly, that it will be reflected with change of sign; but that if reflected at an optically lighter, it will be reflected without change of sign. Thus if the film has refrangibility intermediate between those of the media on the

two sides, the reflexions at its two surfaces will be either both with or both without change of sign; and the conditions of light and darkness should be just opposite to those in the case of the film between two portions of the same medium. This conclusion is verified by experiment.

For the complete investigation of the effect produced by the light coming from the film, we must consider that we have not only to take account of the two rays reflected once at the upper surface and once at the under surface, as C B A, F E D B A,
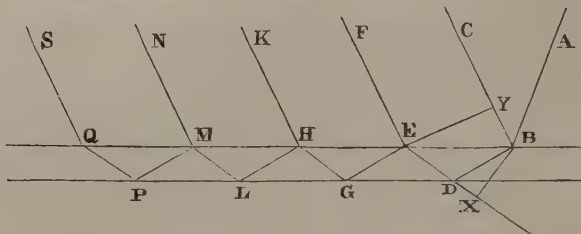


FIG. 174.

but also an indefinite number of rays, such as K H G E D B A, which has undergone three internal reflexions, and so on. And we shall first consider all these rays to be of one definite wave-length, $\lambda$. These rays all combine to produce the resulting ray B A, and the effect will depend on their intensities and their relative retardations. It is clear that the relative retardation between any two consecutive rays, so far as it depends on the optical difference of paths, is the same, and is $2\mu e \cos r$. If there is a further retardation of $\frac{\lambda}{2}$, we may regard this as denoting a reversal in the sign of the amplitude of the vibration. Let us denote the phase-angle of the vibration, at any point in B A, which has come from C B, by $\phi$. Then we may denote the phase-angles of the vibrations at the same point, which have come from the other rays, F E, K H, etc., by $\phi + \delta$, $\phi + 2\delta$, and so on; where—

$$\delta = \frac{2}{\lambda} \cdot 2\mu e \cos r.$$

We have next to consider the relations between the amplitudes of the vibrations arising from the various rays. Suppose that a ray in the outer medium falling on the film gives rise to one reflected at the surface of the film, and one

refracted into the film, whose amplitudes are found from that of the incident ray by multiplying it by the quantities $b$ and $c$. And suppose that a ray in the film gives rise to one reflected back into it, and one refracted into the outer medium, whose amplitudes are found by multiplying that of the first by $e$ and $f$.

Now, suppose the amplitude of any one of the rays incident on the film to be $a$. Then the amplitudes of the various parts along B A, arising from the rays C B, F E, K H, N M, etc., are $ab$, $acef$, $ace^3f$, $ace^5f$, etc.

Certain relations may be found among the coefficients by the following considerations: The ray A B, of amplitude $a$, falling from the outer medium on the material of the film, gives rise to the two along B C, B D, of amplitudes $ab$, $ac$. It is assumed, then, that if two rays of amplitudes $ab$, $ac$ be sent along C B and D B, these will just recombine to pro-



FIG. 175.

duce the ray of amplitude $a$ along B A. But $ab$ along C B gives $ab^2$ along B A and $abc$ along B E, and $ac$ along D B gives $ace$ along B E and $acf$ along B A. Since these four are equivalent to $a$ along B A, we have the two relations—
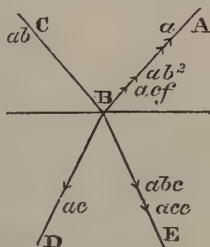
$$abc + ace = 0, \text{ or, } b + e = 0;$$
$$\text{and } ab^2 + acf = a, \text{ or, } b^2 + cf = 1.$$

Thus we can put $e = -b$, and $cf = 1 - b^2$, and write for the amplitudes of the component vibrations along B A, $ab$, $-ab(1 - b^2)$, $-ab^3(1 - b^2)$, etc. And the entire resulting displacement at a point in B A is the sum—

$$ab \sin \phi - ab(1 - b^2) \sin (\phi + \delta) - ab^3(1 - b^2) \sin (\phi + 2\delta) - \cdots$$
$$= ab \sin \phi - ab(1 - b^2)[\sin (\phi + \delta) + b^2 \sin (\phi + 2\delta) + \cdots].$$

The part in square brackets is, by writing the exponential values of the sines—

$$\frac{1}{2i}\{e^{i(\phi+\delta)} + b^2 e^{i(\phi+2\delta)} + \ldots\} - \frac{1}{2i}\{e^{-i(\phi+\delta)} + b^2 e^{-i(\phi+2\delta)} + \ldots\}$$

$$= \frac{e^{i(\phi+\delta)}}{2i} \cdot \frac{1}{1 - b^2 e^{i\delta}} - \frac{e^{-i(\phi+\delta)}}{2i} \cdot \frac{1}{1 - b^2 e^{-i\delta}}$$

$$= \frac{1}{2i} \cdot \frac{e^{i(\phi+\delta)} - e^{-i(\phi+\delta)} - b^2(e^{i\phi} - e^{-i\phi})}{1 - b^2(e^{i\delta} + e^{-i\delta}) + b^4}$$

$$= \frac{\sin (\phi + \delta) - b^2 \sin \phi}{1 - 2b^2 \cos \delta + b^4}.$$

Thus the displacement is—

$$ab \sin \phi - ab(1 - b^2) \cdot \frac{\sin (\phi + \delta) - b^2 \sin \phi}{1 - 2b^2 \cos \delta + b^4}$$

$$= \sin \phi \left\{ ab - ab(1 - b^2) \cdot \frac{\cos \delta - b^2}{1 - 2b^2 \cos \delta + b^4} \right\}$$

$$- \cos \phi \cdot ab(1 - b^2) \cdot \frac{\sin \delta}{1 - 2b^2 \cos \delta + b^4}$$

$$= P \sin \phi + Q \cos \phi, \text{ say.}$$

Then the amplitude of the resulting vibration is $\sqrt{P^2 + Q^2}$; that is—

$$ab\sqrt{\left\{ 1 - \frac{(1 - b^2)(\cos \delta - b^2)}{1 - 2b^2 \cos \delta + b^4} \right\}^2 + \frac{(1 - b^2)^2 \sin^2 \delta}{(1 - 2b^2 \cos \delta + b^4)^2}}.$$

The resulting intensity is proportional to the square of this, or to—

$$2b^2 \cdot \frac{(1 + b^2)^2 (1 - \cos \delta)^2 + (1 - b^2)^2 \sin^2 \delta}{(1 - 2b^2 \cos \delta + b^4)^2},$$

*i.e.* $a^2 b^2 (1 - \cos \delta) \cdot \dfrac{(1 + b^2)^2 (1 - \cos \delta) + (1 - b^2)^2 (1 + \cos \delta)}{(1 - 2b^2 \cos \delta + b^4)^2}$

*i.e.* $a^2 b^2 \cdot 2 \sin^2 \dfrac{\delta}{2} \cdot \dfrac{2}{1 - 2b^2 \cos \delta + b^4},$

*i.e.* $\dfrac{4a^2 b^2 \sin^2 \dfrac{\delta}{2}}{1 - 2b^2 \cos \delta + b^4}.$

The intensity will, therefore, be zero, when $\dfrac{\delta}{2} = n\pi$; that is, when—

$$\frac{2\pi}{\lambda} \cdot 2\mu e \cos r = 2n\pi,$$

or—

$$2\mu e \cos r = n\lambda.$$

The intensity will be a maximum when $\dfrac{\delta}{2}$ is an odd multiple of $\dfrac{\pi}{2}$, or—

$$2e \cos r = n\lambda + \frac{\lambda}{2};$$

as may be seen by writing the above fraction in the form—

$$\frac{4a^2b^2 \sin^2 \frac{\delta}{2}}{(1 - b^2)^2 + 4b^2 \sin^2 \frac{\delta}{2}},$$

or—

$$\frac{4a^2b^2}{(1 - b^2)^2 \csc^2 \frac{\delta}{2} + 4b^2};$$

and observing that for $\frac{\delta}{2} = n\pi + \frac{\pi}{2}$, $\csc \frac{\delta}{2}$ has its least value.

**Colours by Transmitted Light.**—We have hitherto supposed that the light which reaches the film is all reflected light, falling on it from the side on which the eye is, and that no light comes through from the other side. Let us next consider the light transmitted through the film and reaching the eye on the other side. We must deal with this in a similar way, and consider an innumerable assemblage of rays which combine into a single one on leaving the film, the first ray simply undergoing two refractions at the surfaces, the second undergoing two reflexions inside the film as well, the next four reflexions, and so on, as the



FIG. 176.

figure shows. Just as before, the optical difference of path between any two consecutive ones is $2\mu e \cos r$, so that we may write $\delta$ for the difference of phase-angle.
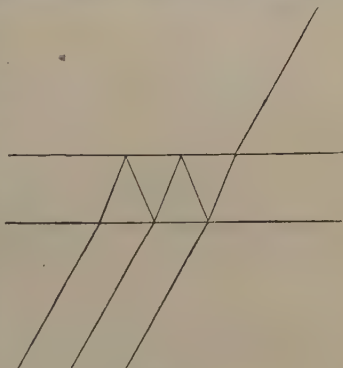
For the amplitudes of the vibrations in the emergent rays, if that of an incident ray is $a$, using the symbols $b$, $c$, $e$, $f$, with the same meanings as before, we have for the amplitudes the values—

$$acf, \quad acfe^2, \quad acfe^4, \text{ etc.,}$$

or—

$$a(1 - b^2), \quad a(1 - b^2)b^2, \quad a(1 - b^2)b^4, \text{ etc.}$$

Calling the phase-angles of the displacement at any point

due to these $\phi$, $\phi + \delta$, $\phi + 2\delta$, etc., the entire displacement is—

$$a(1 - b^2) \sin \phi + a(1 - b^2) b^2 \sin (\phi + \delta) + \ldots$$

Summing this in the same way as before, we get—

$$a(1 - b^2) \cdot \frac{\sin \phi - b^2 \sin (\phi - \delta)}{1 - 2b^2 \cos \delta + b^4}.$$

And for the intensity we have—

$$a^2(1 - b^2)^2 \cdot \frac{(1 - b^2 \cos \delta)^2 + b^4 \sin^2 \delta}{(1 - 2b^2 \cos \delta + b^4)^2} = \frac{a^2(1 - b^2)^2}{1 - 2b^2 \cos \delta + b^4}.$$

The intensity is then a maximum when $\cos \delta = 1$, or $\delta = 2n\pi$; that is—

$$\frac{2\pi\lambda}{\lambda} \cdot 2\mu e \cos r = 2n\pi,$$

or

$$2\mu e \cos r = n\lambda.$$

The intensity is a minimum when $\cos \delta = -1$, or $\delta = 2n\pi + \pi$; that is—

$$2\mu e \cos r = n\lambda + \frac{\lambda}{2}.$$

The maximum value of the intensity is simply $a^2$, showing that in this case all the incident light is transmitted. The minimum value is—

$$\frac{a^2(1 - b^2)^2}{(1 + b^2)^2}.$$

Thus in this case the intensity never becomes zero.

It should be noticed that, for any value of $\delta$, the sum of the intensities of the lights reflected and transmitted, which have the values $\dfrac{4a^2b^2 \sin^2 \dfrac{\delta}{2}}{1 - 2b^2 \cos \delta + b^4}$, and $\dfrac{a^2(1 - b^2)^2}{1 - 2b^2 \cos \delta + b^4}$, is $a^2$, showing that these two portions just account for the whole of the light falling on the film, or that the entire energy in them is just equal to the energy of the incident light.

If, now, in either case it is white light which falls upon the film, the maxima and minima for the various colours will occur at various places, so that the film will show colours. These colours are more strongly marked in the case of reflected light, because in this case, at every point of the film, as a rule, certain colours undergo complete obliteration.

**Newton's Rings.**—Newton produced the colours due to thin plates by this very simple device : Suppose two convex lenses with surfaces of very small curvature, or a convex lens and a piece of plane glass, to be put in contact with each other. There will then be a layer of air between the two, having no thickness just at the point of contact, and with thickness increasing as we pass outward from this point, and such that the thickness is uniform in any circle round this point as centre. If this arrangement is illuminated by mono-chromatic light of wave-length $\lambda$, which is reflected to the eye, a series of black and bright rings will be seen round the point of contact. If $\phi$ is the angle of incidence of the light, it will also be the angle which the rays in the interior air-film make with the normal, and there will be darkness at any point at which the thickness $e$ satisfies the condition

$$2e \cos \phi = n\lambda,$$

where $n$ is an integer.

Suppose that rings are formed in this way, with a spherical surface of radius R in contact with a plane surface. Let O denote the point of contact, and consider the thick-ness, $e$, at a point P. Draw P M N



Fig. 177.

normal to the plane surface, and meeting the sphere of which the given curved surface is a part, in the points M and N. Then—
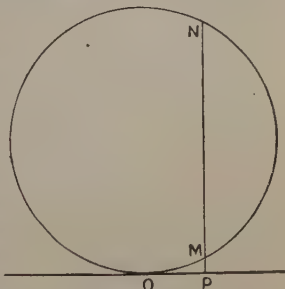
$$PO^2 = PM \cdot PN.$$

But PN is very nearly equal to $2$R. Thus—

$$PO^2 = 2Re.$$

Now, there is darkness at P if—

$$2e \cos \phi = n\lambda,$$

and brightness if—

$$2e \cos \phi = (n + \tfrac{1}{2})\lambda.$$

Thus we get a series of dark rings whose radii are given by—

$$r = \sqrt{R \cdot \sec \phi \cdot n\lambda} \, ;$$

and a series of bright rings whose radii are given by—

$$r = \sqrt{R \cdot \sec \phi \cdot (n + \tfrac{1}{2})\lambda}.$$

Putting $n = 0$ in the formula for the dark rings, we get $r = 0$, and the first ring becomes a black spot at the centre. The rings seen by reflected light then have a black spot in the centre.

If the arrangement is seen by transmitted light, nowhere will there be complete absence of illumination, as we have already seen ; but a series of rings relatively dark and bright will be seen. These are formed similarly to the others, only now the dark rings take the place occupied by the bright when they are seen by reflected light ; and *vice versâ*. The rings seen by transmitted light have a bright spot in the centre.

If the rings are formed by the contact of two spherical surfaces, having radii $R_1$, $R_2$ turned opposite ways, it is clear that, in the above formulæ, instead of R, we must use $R_1 + R_2$.

The rings may be formed, in either case, by means of white light. Then the resulting appearance is the superposition of what would be got from the constituent lights of the various colours. In this case only a few rings will be seen having a dark centre if seen by reflected light, and a bright centre if transmitted, and having variegated edges. As we pass out from the centre, the appearance quickly fades away into practically uniform illumination.

The colours seen in white light, by reflexion, are those due to the successive elimination from white light of lights of various wave-lengths. They were carefully observed by Newton, and given by him in a series which is called Newton's scale of colours. These colours are classed in several orders as follows :—

(1) Black, blue, white, yellow, red ;
(2) Violet, blue, green, yellow, red ;
(3) Purple, blue, green, yellow, red ;
(4) Green, red ;
(5) Greenish-blue, red ;
(6) Greenish-blue, pale red ;
(7) Greenish-blue, reddish-white.

To observe and measure the rings formed in transmitted light, a microscope with a micrometer eye-piece may be used. The arrangement for forming the rings, consisting of lens and plate glass, is placed on the stage of the microscope, and light is thrown upwards to it from the microscope reflector. The rings may then be seen, and their diameters measured by means of the micrometer eye-piece.

To see the rings by reflected light with the microscope, an additional arrangement is necessary. A small thin piece of glass, such as a microscope cover-glass, is placed inside the microscope behind the object-glass. It is inclined to the axis of the instrument at 45°, and there is an aperture in the tube through which light can fall on it, and be reflected out through the object-glass to the ring-forming apparatus, and then back into the microscope, the little reflector not hindering its passage.

If the film is not of air, but of some other substance, such as a drop of water introduced between the lens and the plate glass, the diameters of the rings will be changed. If $\mu$ is the refractive index of this substance, we then have the formula—

$$2\mu e \cos \phi = m\frac{\lambda}{2},$$

where, in reflected light, $m$ is even for a dark place, and odd for a bright one; and in transmitted light the reverse.

The radius of a ring will then be given by—

$$r = \sqrt{R \cdot \frac{1}{\mu} \cdot \sec \phi \cdot m\frac{\lambda}{2}}.$$

On using a drop of liquid, then, in this way, the diameters of the rings will be diminished in the inverse of the ratio of the square roots of the refractive index of the liquid and that of air. Measurements of the rings may be used to determine the refractive index of a liquid.

The diameters of the rings formed with given apparatus increases with the angle of incidence; but the angle of incidence inside the film, if of air, is nearly the same as that on the upper surface of the lens, and, as this angle is increased, more of the incident light will be reflected off the glass. Thus if the diameters are increased by increasing the angle of incidence, it will be at the expense of the brightness of the rings. But by the following device, rings which are broad, and at the same time bright, may be obtained.



FIG. 178.

A glass prism, A B C, is placed with one face in contact with a spherical surface of glass of large radius. Light, of which P Q R S T represents a ray, enters the prism, at a small incidence, by the face A B, and passes into the film of air
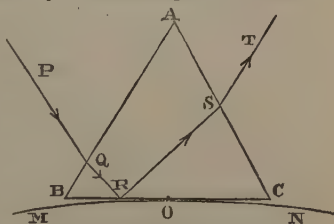
at R, at a considerable angle with the normal at R. Again, on emergence at S, the angle between the ray and the normal at S is small. In this way little light is lost by the reflexions at Q and S, so that the rings are bright; and at the same time, owing to the great obliquity of the rays in the film, they are broad. To produce the rings, sometimes a prism is used with one face made slightly spherical, and this face is rested on a piece of plane glass.

## INTERFERENCES PRODUCED BY THICK PLATES.

Sir David Brewster showed that two thick plates of glass of equal thickness, with a considerable thickness of air between them, could produce interference. For this purpose the plates must be nearly but not quite parallel to each other. Of the
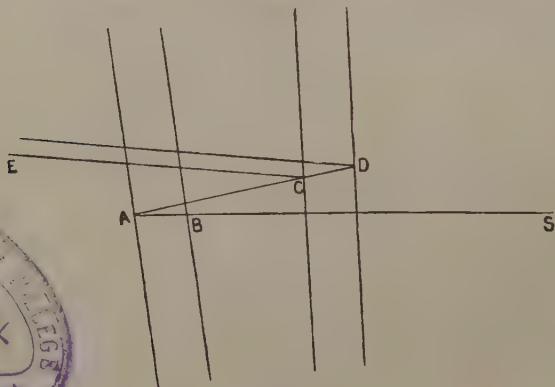


FIG. 179.

light proceeding from a point, S, of the source to an eye at E on the other side of the plates, the greater part goes directly to the eye by transmission through the two plates. But certain of the rays which undergo reflexions at the surfaces may be in a condition to interfere. Thus the two rays which are reflected, one at A and C, and the other at B and D, go on to the eye very close to each other and in parallel directions; also their difference of path (which is acquired by one passing through one plate along a path equal to 2AB, and the other through the other plate along a path equal to 2CD) is a very small quantity. The two rays will consequently be in a condition to interfere on reaching the eye.

Brewster placed his two glass plates at one end of a tube blackened inside, the plates being nearly normal to the tube. The other end was closed by a diaphragm having a small circular aperture. By this means the side light was cut off, but the visible field subtended a large enough angle at the eye for several fringes to be observed. The fringes are parallel to the lines of intersection of the surfaces of the plates; and they diminish in breadth as the angle between the plates is increased. When seen in white light they have coloured edges.

It is not easy to produce these fringes. The glass plates must be of as nearly as possible the same thickness. To ensure this they should be cut from a piece of good plate glass, satisfying pretty well the conditions of a geometrical plate, that is, having uniform thickness. The plates should further be mounted in such a manner that they can be set accurately parallel; if one is fixed, the other should have two adjustments.

This phenomenon is also sometimes called that of **the colours of thick plates.**

**Coloured Rings of Thick Plates, or Diffusion Rings.**—These were discovered by Newton; and Young first gave the explanation of them on the undulatory theory.

A spherical mirror of glass, silvered on the convex so as to reflect on the concave side, is used. Light is allowed to pass through a small aperture in a white screen, held so that the aperture is at the centre of curvature of the mirror. Then a series of coloured rings with a common centre at the aperture is observed. If the aperture is at a little distance from the centre of the mirror, but the screen still passing through the centre, an image of the aperture will be formed by reflexion on the screen, and the rings will now have their common centre midway between the aperture and its image, that is, at the centre of the sphere. There will now be a white ring passing through the aperture and its image. These appearances are due to the diffusion or irregular reflexion of light at the front surface of the mirror; and if this surface is slightly dimmed, as may be done by wetting it with a weak mixture of milk and water, the rings become more distinct.

Suppose O is the centre of the mirror, L O L' M the plane of the screen, and L the aperture, near to O, through which the light comes, L' the image of L, so that L O L' is a straight line, and L'O = OL. A ray L R, after regular refraction and reflexion at R and S of part of it, is partly diffused on emergence at T, so that some diffused light from T reaches

the screen at M. Another ray is diffused at the same point T on entering the glass, and some of the diffused light, after regular reflexion and refraction at S' and R', reaches the screen at M. These two portions of light arriving at M have travelled paths having a small optical difference. And further, the rays have been diffused at the same point of the surface. This is necessary for interference, as has been pointed out by Professor Stokes, because in the act of diffusion a phase-difference is introduced into the vibration, which will be different for different points of the surface. The rays arriving
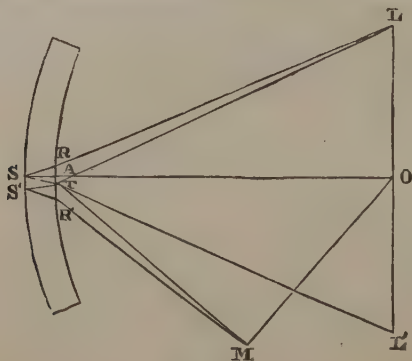


FIG. 180.

at M will thus be in a condition to interfere, and will do so if their path-difference is suitable.

Let us next consider what this path-difference is, on the supposition that L is very near to O. Let the radius O A of the front surface be R; the thickness of the mirror, A S, $= e$; the index of refraction of the glass $= \mu$; $OL = u$; $OM = v$. Let the angles of incidence and refraction at R be $i$ and $r$. Then we have very nearly—

$$LR = \sqrt{R^2 + u^2} = R + \frac{u^2}{2R}.$$

$$RS^2 = e^2 + e^2 \tan^2 r$$

$$= e^2 \left( 1 + \frac{\tan^2 i}{\mu^2} \right) \text{ nearly}$$

$$= e^2 \left( 1 + \frac{u^2}{\mu^2 R^2} \right);$$

$$\therefore RS = e \left( 1 + \frac{u^2}{2\mu^2 R^2} \right).$$

Also ST = RS;

and TM = R + $\frac{v^2}{2R}$.

The path R S T is in glass of refrangibility $\mu$. Hence the optical length of the path L R S T M is—

$$2R + \frac{u^2 + v^2}{2R} + 2\mu e \left( 1 + \frac{u^2}{2\mu^2 R^2} \right).$$

For the ray L T S′ R′ M we notice that regular refraction takes place at R′, so that we may easily show, in a similar manner to the above, that—

$$TS' = S'R' = e \left( 1 + \frac{v^2}{2\mu^2 R'} \right).$$

And the optical length of the path L T S′ R′ M is—

$$2R + \frac{u^2 + v^2}{2R} + 2\mu e \left( 1 + \frac{v^2}{2\mu^2 R^2} \right).$$

The optical difference of paths is thus—

$$\frac{e}{\mu R^2}(u^2 \sim v^2).$$

The phase-difference of the rays arriving at M will thus depend upon $v$, the distance of M from O. In monochromatic light we should thus get a series of bright and dark circles round O ; and in white light, coloured circles. Where $v = u$, that is, for all points on a circle with L L′ as diameter, there is brightness for all colours ; for there is no path-difference in the rays. Hence this is a white circle. At the centre, $v = 0$, there will be a coloured spot, the colour depending on the value of $\frac{eu^2}{\mu R^2}$. Hence the colour will change with the distance of the aperture from the centre.

**Colours of Mixed Plates.**—These were discovered by Young, and are seen when a luminous source is viewed through two glass plates containing between them a very thin layer of two transparent substances intimately mixed up with each other. Brewster obtained them by using a little soap lather between his plates. The explanation of them is that they are due to interferences of two neighbouring rays which traverse, in the space between the glass plates, the two different substances; a difference of phase is thus introduced

between them, and they are in a condition to interfere. If $e$ is the distance between the glasses, and $\mu_1$, $\mu_2$ are the refrangibilities of the two substances, the difference of optical distances traversed by the two rays is, for normal incidence, $e(\mu_1 - \mu_2)$. Thus there will be interference for light of wave-length $\lambda$ whenever—

$$e(\mu_1 - \mu_2) = n\frac{\lambda}{2},$$

where $n$ is an odd number.

### EXAMPLE.

Describe and explain the phenomena presented by the prismatic spectrum of the light reflected from a thin transparent film, not thin enough to show the "colours of thin plates." How and why does the spectrum change as the film gets thinner and thinner until it shows colours? (Lond. B.Sc. Hons., 1884.)

## CHAPTER XV.

### *DOUBLE REFRACTION. POLARIZATION.*

WHEN light is refracted through certain crystalline substances peculiar effects are observed, which we shall now consider.

About the best substance for producing these effects, and



FIG. 181.

that in which they were first discovered, is Iceland spar. This is a crystallized form of carbonate of lime. The form in which it occurs is that of a rhomb or parallelopiped, that is, a six-sided figure, all whose sides are parallelograms. This form is represented in the figure. The angles of any one of the bounding parallelograms are $101° 55'$ and $78° 5'$; and the dihedral angles between the faces of the rhomb are $105° 5'$ and $74° 55'$. The solid angles at A and A′ are, each of them, contained by three equal obtuse angles. If the rhomb has all its edges equal, then the line A A′ is called the crystallographic axis of the crystal: A A′ would then be equally
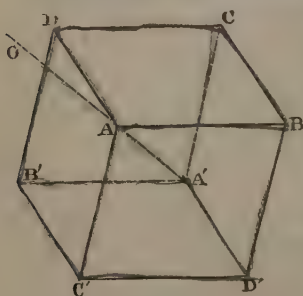
inclined to the three faces meeting in A or A'. In general the axis of the crystal is a straight line given only in direction; and it is any straight line equally inclined to the three faces meeting in a solid angle of the rhomb contained by three obtuse angles. Any plane through the axis is called a **principal plane** of the crystal; and a principal plane perpendicular to any face is called the principal plane of that face.

If a ray of light falls on such a rhomb, it is not refracted into it along one direction, as in the case of a piece of glass, but along two, giving rise to two refracted rays. Thus, if a black mark is made on a sheet of white paper, and then covered by the rhomb, two images of the mark will be seen on looking through the rhomb. Or, again, if a narrow beam of light is falling on a screen, and giving a bright spot, on putting the rhomb in the way of the beam, with a face normal to it, this is broken up into two, so that two spots are observed. The crystal is said to be **doubly refracting**.

In the next place, we have to observe that the light in either of these refracted beams possesses remarkable differences from common light, or the light in the beam before it entered the rhomb. In breaking up the beam of common light by means of the rhomb, the same effect is produced however the rhomb is rotated about the path of the beam—the two emergent beams are of equal intensity. Now, if one of these beams be similarly examined by a second rhomb, this will not be the case.

To show this we may take two similar rhombs, place one of them on the mark on the paper, thus producing with it two images, and place the other on the first. The second will, in general, form two images of each image produced by the first, thus giving rise to four. But if the upper rhomb be rotated over the lower one, this is what is observed: when the faces of the upper rhomb are all parallel to faces of the lower one, so that the principal planes of the contiguous faces are parallel, only two images are seen. As the upper rhomb is rotated from this position, two more images begin to appear, at first very faint, but gradually increasing in distinctness; and the other images diminish in distinctness. And when the upper rhomb has been rotated through a right angle, these images have completely disappeared, and now only the other two are seen. As the rotation continues, the same series of appearances and disappearances occurs in every quarter of a turn.

The same thing may be shown by means of the beam of light passing through the two rhombs to a screen. Or, if we cut off one of the two beams which emerge from the first rhomb, and examine the other alone by means of the second rhomb, the spot of light will appear brightest when all the faces of the rhombs are parallel, and will gradually become fainter, and completely disappear when the second rhomb has been turned through a right angle, appearing again when the rhomb is turned past this position, and growing to maximum brightness when another right angle has been accomplished ; and so on.

From these experiments it is clear that the light in either of the two beams into which a given one is broken up by the crystal of Iceland spar, differs from common light in possessing properties having reference to direction which common light has not. Thus a beam of common light, on entering a rhomb of spar normally to a face, is always broken up into two beams of equal intensity, but a beam which has come from another rhomb is for certain positions of the second rhomb not broken up, and, in general, the intensities of the parts into which it is broken are unequal, and depend upon the position of the second rhomb relatively to the first. The light in either beam coming from a rhomb then possesses some property with regard to direction which must be referred to the manner in which the rhomb is held across its path.

These beams of light are said to be **plane-polarized,** or each is polarized in a certain plane. Their **planes of polarization** are two planes containing the directions of the beams, and respectively parallel and at right angles to the principal section of the face which is normal to the incident beam.

As we shall see, the peculiar property of plane polarized light is that the vibrations of the ether particles constituting it all take place along straight lines in a definite direction. And the plane which is called the plane of polarization of a given beam is, in accordance with the views of Fresnel, the plane at right angles to the direction of vibration of the particles.

We see, then, that by means of a rhomb of Iceland spar we can produce a beam of polarized light, and by means of another rhomb we can determine whether a beam is polarized or not. The rhombs used in this way may be called **polarizer** and **analyzer** respectively.

Huyghens investigated by experiment the laws of refraction

of the two rays in Iceland spar into which a single one is broken up. He found that one of these rays is refracted according to the ordinary laws of refraction, the crystal having for this ray a definite index of refraction, no matter how it is cut or how the incident ray falls on it. This refracted ray is called, in consequence, the **ordinary ray.** The other ray is refracted according to a different law, and not the ordinary law of refraction. The sines of the angles of incidence and refraction do not bear a constant ratio to each other. It is called the **extraordinary ray.** Huyghens accounts for the position of this ray, and shows how to find it in any given case, as follows: We have seen how to find the refracted ray, on the principles of the wave theory, in the case of a ray falling on the surface of glass, by supposing the points of the surface to become centres of secondary disturbances as soon as the wave-front reaches them, these disturbances spreading out with equal velocities in all directions. Thus the disturbance from a given point has at any instant reached all the points on the surface of a sphere traced in the medium. And the refracted wave-front is the plane which at any instant touches all the spheres which the disturbances have reached; and the refracted ray is normal to this plane. Now, in the case of Iceland spar the disturbances which constitute the extraordinary ray travel with different velocities in different directions through the crystal. The surface which a disturbance at a single point in the crystal would reach all the points of in a given time, is not a sphere, but an *oblate spheroid*, or ellipsoid of revolution, whose axis of revolution is the axis of the crystal. This surface is the wave-surface for the extraordinary rays. The wave-surface for the ordinary rays, which is, of course, a sphere, has its diameter equal to the axis of revolution, A A′, of the ellipsoid. Thus, if a disturbance proceeds from



FIG. 182.

the single point O, giving rise to both ordinary and extraordinary waves, at the end of a given time these will have reached the two surfaces of the sphere and spheroid; and
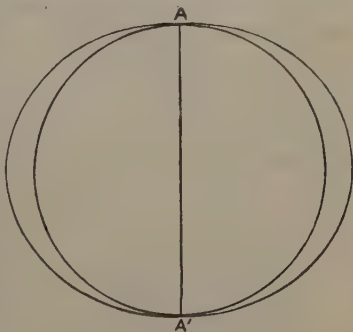
the least of the velocities of the extraordinary waves is equal to the common velocity with which the ordinary waves travel.

The most exact method of showing that the ordinary wave-surface is a sphere, and the ordinary ray refracted according to the ordinary law of refraction, is the following: Several plates of spar cut in various manners from the crystal are all cemented together, and a prism is made with its refracting edge at right angles to all the faces of the plates. If, now, this is used on a spectrometer, each plate will form two images. But one set of images, namely, the ordinary images, will be all quite in coincidence in a vertical line, as if the prism was made out of a single piece.

The construction for the extraordinary refracted ray, and the form of the extraordinary wave-surface on which it depends, may be verified by observations of the incident and refracted rays in various cases.

First, suppose we take the surface of the crystal parallel to the axis, and the plane of incidence normal to the axis, and investigate the positions of the refracted extraordinary rays in this case. This may be done by cutting a prism with its refracting edge parallel to the axis, and using it on a spectrometer. The extraordinary as well as the ordinary image of the slit will be found to follow the ordinary law in this case. Thus we have a constant index of refraction for this case, showing that the velocity of propagation of disturbances normally to the axis is the same in all directions. Or the wave-surface is a surface of revolution about the axis.

The constant index of refraction for extraordinary refracted rays at right angles to the axis is called *the* extraordinary index of refraction of the crystal. But it must be remembered that it only applies to these cases; and the extraordinary ray has, as a rule, no definite index of refraction. The ordinary and extraordinary indices may be denoted by $\mu_o$ and $\mu_e$.

Next we must find by experiment what is the form of the section of the wave-surface by a plane parallel to the axis (that is, by a plane of principal section). Let us suppose that this section is an ellipse, A D A', and that the circular section A C A' of the ordinary wave-surface touches



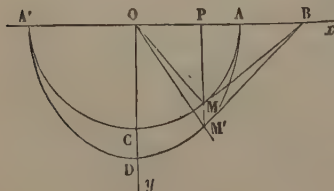Fig. 183.

it at A and A'. Then if P M M' is drawn at right angles to A O A', the tangents at M, M' cut A A' at the same point, B. Thus O M, O M' are ordinary and extraordinary rays corresponding to the same incident ray. And if $r$, $r'$ are the angles of refraction of these rays—

$$\frac{\tan r}{\tan r'} = \frac{PM'}{PM} = \frac{OD}{OC} = \frac{\mu_o}{\mu_e}.$$

To verify this law, Malus experimented as follows : A piece of spar, E F G, cut with faces parallel to the axis, and of
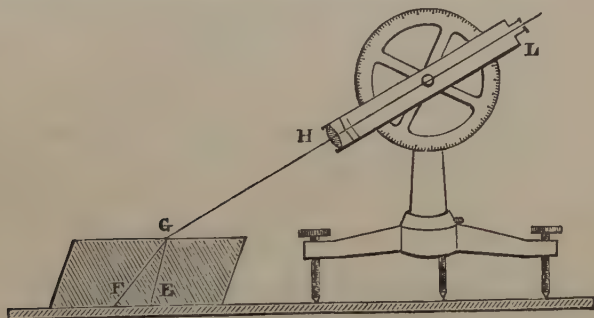


Fig. 184.

considerable thickness, is set on a table with these faces horizontal. H L is a telescope turning over a divided circle, and moving in a plane parallel to the axis of the spar. A scale is placed under the spar, parallel to its axis. Then, on looking through H L, two images of the scale are seen, and the images of two divisions, at E and F, are seen to coincide, being both seen along G H L, these being the ordinary image of one point and the extraordinary of the other. Now, if $e$ is the thickness of the plate—

$$\frac{FE}{e} = \tan r - \tan r'.$$

Also, since the angle between G H L and the normal to the plate is known, $\tan r$ is known. Thus $\tan r'$ is found, and so the ratio $\frac{\tan r}{\tan r'}$. This is found to be equal to $\frac{\mu_o}{\mu_e}$.

These experiments show completely that, within the limits of the errors made in them, the extraordinary wave-surface has the form assigned by Huyghens, namely, that of an oblate

spheroid touching the ordinary wave-surface at the extremities of its axis of revolution.

Mr. Glazebrook has made a series of observations on the extraordinary ray in Iceland spar, and concludes that the construction given by Huyghens for the wave-surface represents its true form within about 1 in 30,000.

Consider next how the extraordinary refracted ray may be found by the help of this wave-surface. Suppose a piece of
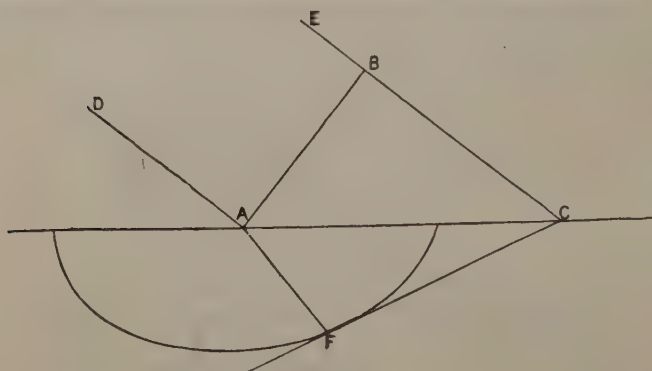


FIG. 185.

Iceland spar to have a plane surface cut in any manner. Let this surface be at right angles to the plane of the paper, and A C its trace on the paper. Suppose a parallel beam to fall on it along D A, E B C; so that A B is a wave-front in this beam. When a point, A, of this wave-front reaches the surface of the crystal, a disturbance starts out into the crystal, and if we consider now only such disturbances as would produce the extraordinary ray, at the end of a given time it would have reached the surface of a spheroid with A as centre. As the various points of the wave-front A B reach the surface of the crystal, such spheroidal disturbances start out. And when the point B of the wave-front has reached the face of the crystal at C, the disturbances starting from the various points of the face between A and C would, if acting separately, have reached the surfaces of spheroids, all similar and similarly situated, but of sizes gradually diminishing from A to C. Now, a plane through C normal to the paper, its trace being C F, will touch all these ellipsoids, and this is the wave-front refracted into the crystal. The refracted *ray* is A F, F being the point where the plane C F touches the spheroid with

centre A; for it is the disturbance which comes from the point A that produces the disturbance at F in the wave-front C F.

It should be noticed that the point F is not necessarily in the plane containing D A and the normal to the refracting surface, that is, the plane of incidence. It will only be so if the axis of the crystal is in this plane. Thus, as a rule, the refracted extraordinary ray is not in the plane of incidence.

**Huyghens's Construction.**—Huyghens has given the following construction, which is very useful in some cases, for determining the position of the ray refracted from one medium into another : In Fig. 191, p. 262, let A C represent the trace of the bounding surface of two media. Let AP denote the direction of the ray in the upper surface incident at A. With centre A describe the two wave-surfaces for the two media corresponding to any interval of time. Let the two curves in the figure denote these. At the point P, where the direction through A of the ray in the upper medium meets the wave-surface for that medium, draw the tangent plane to this surface, cutting the plane of separation of the media in a line through C. This plane will give the position of the wave-front in the first medium for light travelling in the direction A P. Hence the disturbances will all start out simultaneously into the lower medium from the points in the line where this plane meets the plane of separation; and therefore the corresponding plane wave-front in the lower medium is a plane through this line.

Again, if the wave-front in the first medium, parallel to P C, has at any instant just reached A, the time that it takes to reach C is the same as the time that the disturbance takes to travel the distance A P (and in the direction A P) in the first medium. And this is the same as the time that it takes to travel from A to the second wave-surface in the second medium. Hence when the disturbance is starting out from C, it has also just reached the second wave-surface. Thus the tangent plane to this wave-surface, from the line of intersection of the first wave-front and the plane of separation, is the second wave-front. And if Q is the point of contact of the second wave-front with the second wave-surface, A Q is the ray in the second medium.

If either of the media is doubly refracting, there will be two sheets of the wave-surface for it; and this construction will give, in general, two refracted wave-fronts and two refracted rays.

Let us return now to the polarization of the two beams

of light into which a given one is split up by passing it through a rhomb of spar. These two beams we may now call the ordinary and the extraordinary beam. As a rule, each of these, by passing through a second rhomb, will give rise to two beams. If the second rhomb, has its principal plane parallel to that of the first, the ordinary beam gives only an ordinary ; and on rotating the second rhomb an extraordinary beam begins to appear, and the ordinary to fade away ; and when the principal planes are at right angles, the ordinary beam gives only an extraordinary. If we similarly examine the extraordinary beam with the second rhomb, we shall find that it gives only an ordinary when the principal planes are at right angles, and only an extraordinary when they are parallel, and both in intermediate positions. The two beams are seen, on examining them with the second rhomb, to be exactly alike, only that each is like the other turned about the direction of its propagation through a right angle. The ordinary beam is said to be polarized in the principal plane of the crystal in which it lies, or to have this for its plane of polarization ; the extraordinary beam is polarized in a plane at right angles to the principal plane.

If a plane polarized beam of light is produced in any manner, we may say that its plane of polarization is the principal plane of a rhomb of spar in which lies the ordinary beam produced by the rhomb which exactly coincides both in direction and in polarization with the given beam. But, at the same time, as plane polarized light may be produced by other methods, other practical definitions may be given of plane of polarization.

On account of the optical properties of the crystallographic axis of the spar, it is also called its **optic axis.**

A beam of light may traverse the crystal in such a manner that both ordinary and extraordinary coincide in direction with the axis. In this case there is no bifurcation of the beam, and it emerges as ordinary light.

There are many other crystals producing effects on light similar to those of Iceland spar ; that is, dividing a beam by refraction into an ordinary and an extraordinary polarized beam. These are divided into two classes, according as the ordinary or the extraordinary index is the greater.

Crystals in which $\mu_e$ is less than $\mu_o$ are called **negative,** or **repulsive.**

Crystals in which $\mu_e$ is greater than $\mu_o$ are called **positive,** or **attractive.**

Those in the first class are called repulsive because the extraordinary seems to be repelled by the optic axis; similarly for the name attractive.

Iceland spar and quartz are types of negative and positive crystals respectively. The two indices of refraction for spar and quartz for sodium light, as determined by Rudberg, are given as follows :—

|  |  | Spar. | | Quartz. |
|---|---|---|---|---|
| Ordinary | ... ... | 1·65850 | ... | 1·54418 |
| Extraordinary | ... | 1·48635 | ... | 1·55328 |

It is frequently required to have a single beam of plane polarized light. We have seen how a rhomb of Iceland spar may be used to produce two beams polarized in planes at right angles to each other. If when a single beam is required we use a rhomb and stop off one of the beams by means of an opaque screen, in order to have a pretty large beam a very large rhomb would be required. This would be both expensive to procure and inconvenient to use. One of the best devices for getting a beam of plane polarized light is Nicol's prism (commonly called a Nicol).

**Nicol's Prism.**—To make the prism, a rhomb of Iceland spar is taken with its faces those formed by natural cleavage. The rhomb should have two end faces, each having equal sides, and the other four edges longer than these. The plane through the two of these four which are blunt edges, and which is consequently at right angles to the end faces, is a principal plane of the rhomb. The rhomb is cut in two by a plane perpendicular to this principal plane, and at an angle of about 22° with the blunt edges. The faces thus formed are polished and cemented together again with Canada balsam, a material having a refractive index intermediate between the ordinary and extraordinary indices of the spar. The end faces are then cut down to form faces still at right angles to the principal plane, and about at right angles to the plane of section.

The following geometrical construction, based on Huyghens's construction for refracted rays, and taken from Jamin's " Cours de Physique," explains the action of the Nicol's prism.

Let A B D C represent the plane through the two blunt edges, A D the trace on the paper of the plane of section, A X the axis of the rhomb. Suppose the incident rays to be in the principal plane A B D C. Then they will remain in this plane. With centre A describe the wave-surfaces for the crystal, Canada balsam, and air. The traces of these on

the paper will be three circles and an ellipse.  Let the circles
meet A D in the points E, F, G.  The circle through F must
cut the ellipse, because the index of the balsam lies between



FIG. 186.

the ordinary and extraordinary indices of the spar.   Consider
an extraordinary ray in A C D.  To find its direction on entering
the balsam, from A draw a radius vector to the ellipse parallel
to it; from the extremity of this draw a tangent to the ellipse;
and from the point where this meets A D draw a tangent to
the circle through F.  The straight line joining A to the point
of contact gives the direction of the ray in the balsam.   But
if no tangent can be drawn to the circle through F, there is no
refraction into the balsam, but the ray is totally reflected at its
surface.   Thus if F H is drawn to touch the ellipse, A H is
the limiting extraordinary ray that is refracted into the balsam;
all extraordinary rays between A H and A D will be totally
reflected.   In the same way, if F K is drawn to touch the circle

through E, all ordinary rays between A K and A D will be totally reflected at the surface of the balsam.

We must next consider the paths of the rays before they entered A C D. Let F H and F K produced meet C A in L and M. Draw L N, M O tangents to the circle through G. Then A N, A O are the directions of the limiting extraordinary and ordinary rays in air. Rays of natural light incident on the face A C, in the principal plane, in any directions intermediate between A N and A O, will give rise to extraordinary rays that will pass through the balsam, and therefore through the whole prism; but the ordinary rays produced will be totally reflected at the surface of the balsam.

**Foucault's Prism** differs from Nicol's in having a layer of air between the two halves instead of the balsam. With this arrangement the action is in principle the same. The prism is shorter and less expensive than Nicol's. But the advantage of Nicol's is that the beam which it transmits is brighter, because the index of the balsam is nearly the same as that of any extraordinary ray entering it, so that little light is lost by reflexion.

It should be carefully remembered that in these prisms it is the extraordinary beam that is let through, and therefore the light obtained is polarized at right angles to the principal plane, that is, in the plane containing the two long acute edges of the prism. This is the **polarizing plane** of the prism.

The prism may be used to obtain from a beam of natural light a beam of plane polarized light, and then it is a *polarizer*. Or it may be used to examine the polarization of a beam. If a beam examined by the prism is totally plane polarized, then on rotating the prism round a line parallel to its long edges the emergent beam is greatest when the plane of polarization of the original beam coincides with the polarizing plane of the prism, and gradually diminishes in intensity and becomes extinguished when these two planes are at right angles. Thus the plane of polarization of the beam can be found. Or if the beam is partially polarized in some plane, that is, if it consists partly of plane polarized light and partly of natural light, by rotating the prism and noting the positions of maximum and minimum intensity of the emergent beam, the plane of polarization can be found. The prism is then an *analyzer*.

**Polarization by Tourmaline.**—The crystal tourmaline is doubly refracting, producing an ordinary and an extraordinary beam. But it absorbs these two beams very differently.

Even about 1 millimetre thickness of it allows practically none of the ordinary ray to pass. So that when a beam of natural light falls on a plate of tourmaline cut parallel to the axis, a beam of plane polarized light is obtained. This affords a very ready means of obtaining plane polarized light. Tourmaline produces some absorption of the extraordinary beam as well; so that with an incident beam of moderate intensity only a faint polarized beam is obtained. It is thus necessary to use with it a strong source of light.

**Rochon's Prism.—** It is sometimes useful to be able to divide a beam of light into two in different directions, or to form two images of an object. Rochon's double-image prism is a means of doing this. A B C, A B D are two prisms of quartz or Iceland spar, with refracting angles at A and B equal. In A C B the axis is at right angles to the face A C, and in A B D the axis is parallel to the edges (perpendicular to the plane of the figure). Let us suppose the prism made of quartz, in which the refractive index for an extraordinary ray is greater than the ordinary index. If a ray falls on the face A C normally at C, it goes along P Q without deviation and without division. At Q, however, on entering the prism A B D, it is doubly refracted. The ordinary ray goes along Q R S without deviation. The extraordinary ray is bent towards the normal to the inter-face A B at Q, since its velocity in A B D is less than in A B C, or its index of refraction from A B C to A B D is greater than unity. Again, this ray is bent away from the normal on emerging at the face B D, so that its path is P Q T U. Thus the ray of natural light falling normally on the face A C is broken up into two polarized rays, having a definite angle between them.
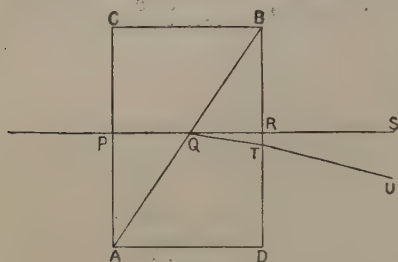


FIG. 187.

**Wollaston's Prism** differs from Rochon's in having the first prism, A B C, cut with its axis parallel to A C. In this case both rays are deviated by equal amounts and in opposite senses; and the angular separation obtained is double that in the last case.

Rochon's prism is used with a telescope to determine the angle subtended by two distant points, as, for instance, to find

the angular diameter of a heavenly body. If O is the object-glass of the telescope, and F its principal focus, the prism is placed in a position, P, so as to form two contiguous images,
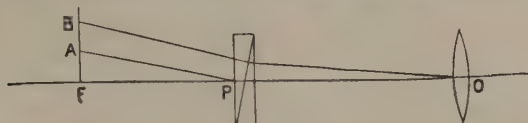


FIG. 188.

F A, A B, of the distant body. The highest point of the undeviated image and the lowest of the deviated image coincide at A. Now, let $OF = f$, $PF = p$. Let $\delta$ be the deviation, A P F, produced by the prism, and let the required angle, which is equal to A O F, be $\theta$. Then—

$$AF = f \tan \theta = p \tan \delta.$$
$$\therefore \tan \theta = \frac{p \tan \delta}{f}.$$

A scale may be fixed to the telescope for determining $p$; and the quantity $\dfrac{\tan \delta}{f}$ may be found by making an observation on a body for which the value of $\theta$ is known (as on a body of known height at a known distance), and observing the corresponding value of $p$. Having thus found the value of $\dfrac{\tan \delta}{f}$, which is the constant of the arrangement, by simply reading off $p$ in any case, and multiplying by this constant, we find the value of $\tan \theta$, that is, practically, the value of $\theta$.

**Interference of Polarized Light.**—This was investigated by Fresnel and Arago. They found that two beams of light, coming originally from the same source, if plane polarized in parallel planes, will interfere, but if in planes at right angles to each other, they will not interfere. These results were found to be the same, whether the two beams were both ordinary or both extraordinary beams from Iceland spar, or whether one was ordinary and the other extraordinary.

These results are what would be expected from conclusions already drawn with regard to the nature of light, and polarized light in particular. The phenomenon of interference with natural light indicates that light consists of vibrations, but shows us nothing about the direction of the vibration; they might even be longitudinal, like sound-vibrations, as far as this phenomenon goes. But the experiments on polarized

light made with rhombs of spar show that such light possesses different properties with regard to two planes through its line of propagation and at right angles to each other. Hence we infer that in this case the vibrations are in some definite direction at right angles to the line of propagation; and that for light in general the vibrations are at right angles to the line of propagation. The experiments on interference of polarized light may be regarded as confirming these views, and showing that there is no longitudinal vibration or component of vibration concerned in the propagation of light, for in that case interference could be obtained between two beams from the same source polarized in planes at right angles to each other.

## CHAPTER XVI.

### *FRESNEL'S THEORY OF DOUBLE REFRACTION.*

WHEN a beam of natural light is broken up into two beams plane polarized in planes at right angles, as by a rhomb of Iceland spar, the intensity of each of these beams is half that of the given one. We must suppose that the displacement in the direction of vibration of an ether particle in a polarized beam is the resolute in that direction of the displacement it would have received if the original beam had gone on unchanged through the path of the polarized one. Thus two corresponding particles in the polarized beams have for displacements the rectangular components of a displacement in the original beam. And the vibrations in the polarized beams compound into the vibrations in the original beam. Now, the vibrations in a beam of natural light are equally in all directions across the line of propagation. Thus the mean amplitude of vibrations is the same in each polarized beam; the energy of the vibrations is equally divided between them, and each is of half the intensity of the original beam.

Suppose a plane polarized beam to be broken up into two others polarized in planes making angles $a$ and $90° - a$ with the plane of polarization of the original beam. Then the direction of vibration in the first makes angles $a$ and $90° - a$ with those in the other two. And we can resolve a displacement in the given beam into two in the resultant beams. If O A denotes the amplitude of vibration in the given beam, and $Ox$, $Oy$ are the directions of vibration in the resulting beams,

O M and O N, the resolutes of O A, denote the amplitudes in these beams.

$$OM = OA \cos a; \quad ON = OA \sin a.$$

Now, the intensity of a plane polarized beam is proportional to the square of the amplitude, for the energy of vibration is proportional to the square of the amplitude. Thus if the intensity of the given beam is I, the intensities of the other two are $I \cos^2 a$ and $I \sin^2 a$.

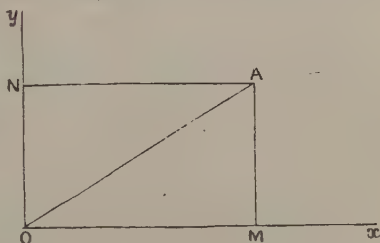These results may be verified by reference to the beams into which a plane polarized beam is



FIG. 189.

broken up by a rhomb of spar. These two beams would be found to have intensities equal to $I \cos^2 a$ and $I \sin^2 a$, except so far as some light is lost by reflexion on entering and emerging from the spar.

The theory which is usually accepted as accounting for the action of crystals on light, in dividing a beam, and in producing polarization, is that of Fresnel. Of this theory we shall now give some account.

In the case of empty space, or any ordinary transparent medium, such as glass, if an ether particle be displaced from its position of rest in any direction whatever, the force tending to restore it to its old position is directed towards this position, and is the same for a given displacement, no matter what the direction of the displacement. Such a medium has no properties depending on direction; at a given point its properties are the same in all directions. It is called an **isotropic** medium. A substance which has properties depending on direction is called **anisotropic**. For instance, a substance of lanellar structure, such as mica, is anisotropic, so is any substance which allows heat or electricity to pass through it more readily in some directions than in others. This state of things in a body must be carefully distinguished from heterogeneity. A substance is homogeneous if it possesses the same properties at all points, and heterogeneous if its properties vary from point to point. A substance is anisotropic if its properties at a given point vary with the direction chosen through that point. A substance may be homogeneous and at the same time

anisotropic, as a crystalline body would be if of perfectly regular structure. Any element of the body would be exactly like any other element; but in any element the properties would depend on the direction, for instance, the crystal could anywhere be more easily broken across certain definite directions than across others.

A crystal of transparent substance is a body possessing properties that render it optically anisotropic, as a rule. It behaves towards light differently in different directions. For instance, the refractive index of a crystal is generally not constant, but depends on the direction of the ray of light traversing it. Thus the velocity of light through a crystal is different for different directions. The simplest method of explaining this peculiarity, in accordance with the wave theory of light, is by supposing that when an ether particle is displaced in a crystal the force called into play depends upon the direction of the displacement, unlike what happens in the case of glass or such a medium.

The difference between the two cases may be illustrated by reference to a mechanical analogy. Imagine a stiff metal rod with one end clamped in a vice, so that by springing the other end it may be made to vibrate. Now, suppose, first, that the rod is of circular section, so that, however the free end is displaced, the force called into play is towards the old position and proportional to the displacement. If, then, the rod is displaced in any manner and left to vibrate, it will perform vibrations whose period is independent of the direction and of the extent of the displacement. These vibrations of the free end would be along a straight line through the equilibrium or mean position. This end may, however, have a motion impressed upon it so that it does not move along such a straight line. Being then under the action of a force which is always directed to the mean position and proportional to the displacement from this position, it follows from the principles of mechanics that it will continually describe an ellipse about this position as centre. This is, in general, the way in which a particle will move when under the action of a restoring force proportional to its displacement, and directed towards its equilibrium position. When the path is a straight line, we may regard this as a particular case of the ellipse, one of the axes vanishing.

This represents the vibrations of an ether particle in an isotropic medium. The particle vibrates in an ellipse. The path is continually changing, but a large number of vibrations

take place before there is appreciable change in it, as the experiments of Fizeau and Foucault and others who have counted interference bands, have shown. Thus in an isotropic medium, vibration of an ether particle in any given elliptical path can be maintained.

Next let us suppose that the vibrating metal rod is flatter one way than the other, say that it is of rectangular section, the sides of the section being *a* and *b*, *a* being longer than *b*. Then if the rod be displaced in the direction of either *a* or *b*, and let go, it will perform vibrations along a straight line through its mean position; but the periods of the vibrations will be different in the two cases, for the restoring force for a given displacement will be greater when the displacement is in the direction of *a* than when it is in the direction of *b*. For a displacement in any other direction, the restoring force will not be in the direction of the displacement, but will more nearly coincide with *a*. An elliptic vibration cannot in this case be maintained. The only simple vibration which can be maintained is a rectilinear one in the direction of either *a* or *b*.

This represents what takes place when light travels through a crystal. In general, when an ether particle is displaced in a given plane wave-front, there are only two directions of displacement producing restoring forces tending to the mean position; and these two directions in the given wave-front are at right angles to each other. Thus for a plane wave-front travelling through the crystal, and normal to a given direction, the only states of vibration of the ether particles are along straight lines parallel to one or other of two directions in the wave-front, and at right angles to each other. These are called the **singular directions** in the wave-front. However, even in the case of a displacement in one of these directions, the resulting force does not, as a rule, tend to the equilibrium position, but will be such that its component in the wave-front does so, and it will have another component along the normal to the wave-front. This latter component can have no effect on the propagation of disturbances through the ether, because the ether is regarded as being incompressible; and if forces acting normally to the wave-front can transmit disturbances, these must be longitudinal, as in the case of sound-waves, and must consist of compressions and dilatations of the medium. Thus the directions of vibration for a given wave-front in the crystal are such that the force called into play by a displacement

taking place along either of them is in the plane containing
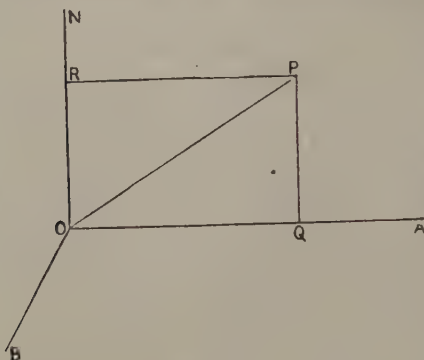that displacement and the normal to the wave-front.    Let us



denote by O A B the
plane of the wave-
front, O A and O B
being the directions
in it in which vibra-
tions can be main-
tained.   Let O N be
normal to the wave-
front.   Suppose that
a displacement along
O A produces a force
represented by P O ;
P O must be in the
plane A O N.   Resolve
P O into its compo-
nents Q O and R O,

Fig 190.

along O A and O N.   Q O is the force which is effective in
propagating the disturbances in the direction O A.

Thus a beam of light travelling through a crystal will,
in general, have this peculiarity distinguishing it from ordinary
light, that the vibrations in the wave-front are always along
straight lines parallel to a definite direction.    And if an
ordinary beam of light falls on the crystal, it will, on entering
it, be divided into two beams, such that the vibrations in the
wave-front of either are parallel to one of the singular direc-
tions in that wave-front.   Such a beam of light is called a
**plane polarized beam**, the vibrations being all parallel to a
plane containing the normal to the wave-front.   What is called
the **plane of polarization** of the beam, however, in accord-
ance with Fresnel's views, is a plane through the normal to the
wave-front, and *at right angles to* the direction of vibration.

In some crystals, such as those of cubical form, this
breaking up of the beam is not produced, these crystals being,
for optical purposes, isotropic.    And in any crystal which
does divide a beam, there is always one, and sometimes two,
directions in which a beam of ordinary light may travel without
being divided.   These directions are called optic axes of the
crystal.   And a crystal is called uniaxal or biaxal according as
it has one or two optic axes.

Whenever a wave is divided by a crystal into two plane
polarized waves, these travel through the crystal with unequal
velocities.   For the forces acting on an ether particle to

restore it to its equilibrium position, when it is displaced by a given amount from that position, are different in the two waves; that is, these restoring forces have different values for the two directions in which the vibrations take place. In other words, the effective elasticities of the ether for displacements in the two directions of vibration are different. Now, if a disturbance is transmitted through an elastic medium, the velocity, V, of the disturbance is connected with the density, $d$, of the medium and its elasticity, $e$, corresponding to the given sort of disturbance, by the relation—

$$V = \sqrt{\frac{e}{d}}$$

And, the density of the ether being the same, whichever of the waves through the crystal we are considering, it follows that the velocities of the two waves are proportional to the square roots of the elasticities for the corresponding displacements.

From what has been said, it will be seen that along the same direction, that is, with a common normal, two different wave-fronts can be propagated in a crystal. These will have different velocities, and will be such that their vibrations are at right angles to each other. Therefore the planes of polarization of the two waves, which can travel along the same direction, are at right angles to each other.

Now, the direction in which a wave-front travels in a medium, when the incident wave-front is given, depends upon the velocity with which it travels; for the refractive index of the medium is the ratio of the velocity of the wave in air to that in the medium. Hence, for the two wave-fronts into which the incident one is broken up, the crystal will behave as if it had different refrangibilities, and they will travel in different directions. Such a crystal is therefore called **doubly refracting**.

Instead of a wave-front of disturbances falling on a crystal, let us suppose that a disturbance is created at a single point. In the case of an isotropic homogeneous medium, this would travel out with equal velocities in all directions, and at the end of a given time would have reached all the points on the surface of a sphere. In the case of the anisotropic crystal, however, the velocities of propagation of the disturbances are different in different directions; and at the end of a given time the disturbance will have reached a certain surface. This is called the **wave-surface** of the crystal. When a wave-front is travelling through the crystal, if we describe the wave-surfaces for a given interval of time, $t$, about

all the points of the wave-front as centres, these will all be touched by a plane parallel to the given front, this plane being their envelope. This plane is the region which the disturbances in the given wave-front will have reached in the time $t$, or is the new position of the wave-front. Also, since a wavefront in the given position is capable of travelling with either of two velocities, it follows that the wave-surfaces described must have two plane envelopes; thus each of them must consist of two sheets.

If we know the form of the wave-surface, we can determine the plane wave-fronts inside a crystal into which a given incident one is broken up. Suppose the wave-front A B in
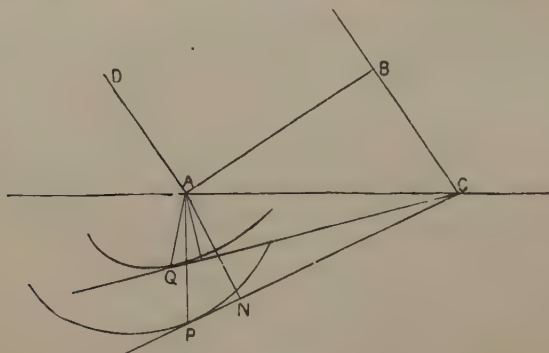


FIG. 191.

air incident on the plane face A C of a crystal. Draw the wave-surface with centre A, corresponding to the time taken by the disturbance at B to reach C. At the time when the wave has reached C, the disturbances from the various points along A C will have reached wave-surfaces enveloped by two planes, C P and C Q. These planes, then, are the two wavefronts into which the given one has been broken up. Draw A N normal to the plane C P. A N is the direction in which the wave C P travels.

The direction of the *ray*, however, which passes through A is not A N, but A P, the straight line drawn from A to the point at which the plane C P meets the wave-surface having A as centre; for of the disturbances in the wave-front C P, that at P, a point on the wave-surface with A as centre, is created by the disturbance at A. If a screen be placed on the surface of the crystal, having a small aperture at A,

so as to admit only a very limited part of the wave into the crystal, the light admitted would travel along A P. Thus the ray D A undergoes double refraction, giving rise to the two rays A P and A Q.

Let us consider next the form of the wave-surface found by Fresnel for doubly refracting crystals in general. To do this we must first consider another surface relating to the crystal, namely, **the ellipsoid of elasticity.**

In a crystal, in general, if an ether particle is displaced, the restoring force is not along the line of displacement. But there are three directions mutually at right angles, such that for displacements along them the restoring forces are along them. These are called the crystallographic axes of the crystal. For a unit displacement along these three, suppose the restoring forces are A, B, C. These quantities are in general all different. In an isotropic medium they are all equal, and then unit displacement in any direction produces the same restoring force, which acts along that direction. For a uniaxal crystal two of these three quantities are equal, say $B = C$. Then unit displacement along any direction in the plane of the two directions corresponding to B and C produces the same restoring force, which acts along the direction of the displacement. In this case the crystal has only one crystallographic axis. Consider the most general case, where A, B, C are all unequal. Now, we must remember that the force produced by any displacement is the resultant of the forces produced by the components of that displacement; and that the force produced is proportional to the displacement. Imagine, then, a unit displacement whose components along the three axes are $\xi$, $\eta$, $\zeta$. The restoring forces parallel to the axes are—

$$A\xi, \ B\eta,$$

Now, the cosine of the angle between the first axis and the displacement is $\xi$. Hence, resolving these forces along the line of the displacement, the restoring force *along this line* is—

$$A\xi^2 + B\eta^2 + C\zeta^2.$$

Now, suppose from a point O we draw three straight lines parallel to the axes of the crystal, and lay off along them lengths whose squares are inversely proportional to A, B, C, and construct an ellipsoid with O as centre and these lengths as semi-axes. The equation of the ellipsoid will be—

$$Ax^2 + By^2 + Cz^2 = K;$$

and the above expression for the restoring force along the line of displacement will be inversely proportional to the square of the radius vector of the ellipsoid in the direction of the displacement.

The ellipsoid thus constructed is called the **ellipsoid of elasticity** of the crystal; and it has the property that a displacement of an ether particle by a given amount in any direction produces a force whose resolute in that direction is inversely proportional to the square of the corresponding radius vector.

That is, if $r$ is the radius vector, the effective elasticity for wave-propagation is proportional to $\frac{1}{r^2}$. Or, since the velocity of propagation of a wave is proportional to the square root of the elasticity, if a plane wave has its vibrations in the direction of $r$, the velocity with which it travels normally to the wave-front is proportional to $\frac{1}{r}$. This is a most important property of the elasticity ellipsoid.

Now, imagine a plane wave-front whose normal is O N. Let this wave-front cut the ellipsoid with O as centre in the ellipse A B C. Suppose the vibrations in the wave-front are along A C, so that O A is $r$. Then for this wave to be permanently propagated, the force produced by the displacement must be in the plane A O N. Now, by the geometry of the ellipsoid, this force, whose components are A$\xi$, B$\eta$, C$\zeta$, is in a plane normal to D B, the diameter of A B C D conjugate to O A. Thus the diameters A C, B D must be at right angles, or they are the axes of the ellipse A B C D. Hence, with the given wave-front, there can only be vibrations along O A or O B, the axes of the section by it of the elasticity ellipsoid. And the velocities with which the corresponding waves travel are inversely proportional to O A and to O B.
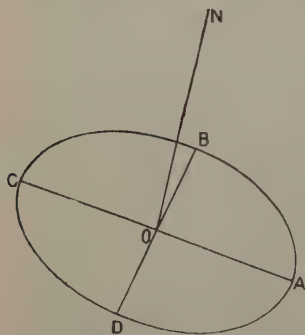


FIG. 192.

This shows that in a given direction two plane polarized beams can be propagated; their planes of polarization being at right angles to each other, and their velocities different.

Now, if we take two points, P, Q, on O N, and draw planes

through them at right angles to O N and inversely proportional to O A, O B, they will be the positions at a certain instant of the two wave-fronts which can travel along O N ; consequently they will touch the wave-surface with centre O for that instant ; that is, the surface which would be reached at that instant by a disturbance starting from O. Hence this surface is found by drawing all the sections of the elasticity ellipsoid, laying off on their normals through O distances inversely proportional to their axes, and drawing planes at these distances from the sections. The surface is touched by all these planes, or is the envelope of them.

The surface is symmetrical about the three rectangular planes containing the axes of the crystal two and two. The
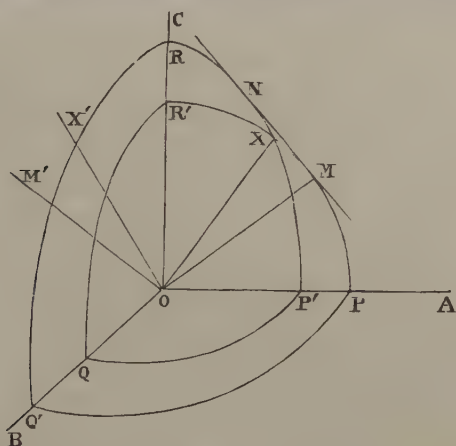


FIG. 193.

figure represents the sections of one-eighth of it by these three planes. It consists of two sheets, an inner and an outer one ; these meet at four points, such as the one, X, which is shown.

There are two directions in the crystal for each of which there is only one velocity for the wave-front, and the vibrations may have any direction in either of these wave-fronts. For there are two sections through the centre of the elasticity ellipsoid which are circles. And any two rectangular diameters of one of these may be taken as axes. Hence a plane wave parallel to one of these circles with vibrations in any direction may be propagated. Also, since all diameters of the circle

are equal, the velocities of propagation of all such waves, whatever be the direction of vibration, will be the same.

The directions of propagation of these two waves, or the normals to the circular sections of the elasticity ellipsoid, are called the **optic axes** of the crystal, or the **axes of single-wave velocity**.

It is clear that only one plane normal to an optic axis can be drawn to touch the wave-surface on one side of the centre. The trace of such a plane on the plane A O C is shown by M N. The lines O M, O M′ normal to this and a similar plane are the optic axes.

We have seen that the *ray* corresponding to a given wave-front is found by joining the centre of the wave-surface to the point at which this is touched by the wave-front, and it must not be confused with the normal to the wave-front, or direction in which the wave-front is travelling. Now, a straight line through O will, in general, cut the surface in two points. Thus along the same direction two rays can be propagated with two different velocities. They correspond to two different wave-fronts, namely, the tangent planes at the points in which they meet the surface. And the vibrations in them are in two different directions, that is, they are polarized in different planes.

But a straight line drawn from O to X will only meet the surface in one point, X. And a ray along this line can only have one velocity. Corresponding to this ray we can have an infinite number of wave-fronts. For at the point X, where the two sheets meet, the surface has the form of a double hollow cone, a depression in the outer sheet meeting a protuberance in the inner at this point. An infinite number of planes can be drawn to touch the surface at X, just as an infinite number of planes can be drawn all touching a double cone and passing through its vertex.

Now, any of these planes touching the surface at X can be taken as wave-front, and will lead to the ray O X. O X produced backward passes through another point of the surface like X, and there is another such line, O X′, passing through two points like X. The lines O X, O X′ are called **the axes of single-ray velocity.** They must be carefully distinguished from the **optic axes**, or **axes of single-wave velocity**, although they are generally very close to these. The ray of light along O X may be polarized in a variety of different planes. For the plane of polarization will depend upon the position of the corresponding wave-front, and we see that this may have an infinite number of positions.

There is another remarkable property of the wave-surface, indicating peculiar actions of the light passing through the crystal. The wave-front M N, corresponding to an optic axis, does not merely touch the surface in two points, but all round a circle. Hence corresponding to this wave-front there is an infinite number of rays in the crystal, namely, all those from O to the various points of the circle of contact of M N. And the disturbances reaching any point of this circle from O, just as those reaching any other point of the wave-surface, are in a definite direction. Thus the rays in the hollow cone of light O M N, corresponding to the wave-front M N, are polarized rays.

These two properties of the wave-surface, namely, its possessing a single ray corresponding to an assemblage of wave-fronts, and a single wave-front corresponding to an assemblage of rays, lead to two remarkable phenomena, which we shall now consider.

**External Conical Refraction.**—Suppose O O' is an axis of single-ray velocity in a biaxal crystal A B C D. If a plane polarized ray travels along O O' in the crystal, there will be a definite wave-front corresponding to it, one of the infinite number of tangent-planes at X to the wave-surface, the one corresponding to the given ray depending on the plane in which the ray is polarized. To find the emergent ray we must take the line in which the wave-front meets the face C D of
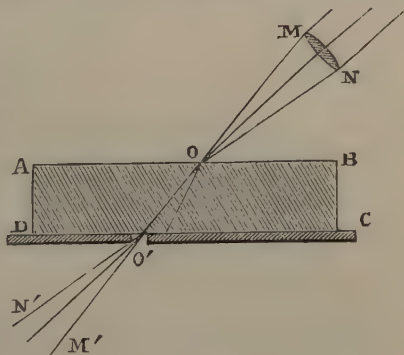


Fig. 194.

the crystal, and through it draw the tangent-plane to the spherical wave-surface for air with centre O'. Then the emergent ray is along the line joining O' to the point of contact. Thus if we have rays along O O' polarized in all planes through O O', that is, having for wave-fronts all the tangent-planes at X, the emergent rays will have an infinite variety of directions, and will form a hollow cone, O M' N'. Now, the emergent ray O' M' is plane polarized. If the crystal is a plate, and M O is parallel to O' M', a ray along M O, polarized in the same plane

as the ray O' M', will give rise to the ray O O' in the crystal and the emergent ray O' M'. And a ray of natural light along M O will give rise to these and a plane polarized ray in the crystal in some other direction. That is, this common one ray will be doubly refracted, one ray going along O O'. If, then, a converging beam of light be concentrated to O, this beam including rays in all the directions such as O' M' formed for the emergent rays from O, and if a screen be placed on the other side of the crystal with an aperture at O', a hollow cone of plane polarized rays will proceed from O'.

**Internal Conical Refraction.**—Suppose that, with such a plate of crystal as that just used, an incident ray, L O, is
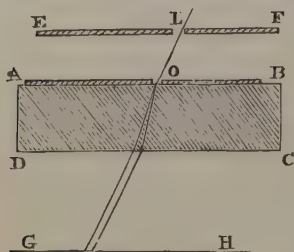


Fig. 195.

produced, by means of apertures in screens A B, E F, in such a direction as to give the refracted wave-front which touches the wave-surface along a circle. Then, if there is in L O light polarized in all planes through L O, or if the light is natural light, there will be rays refracted from O to all the points of contact of this wave-front with the wave-surface. That is, a hollow cone of light will proceed from O into the crystal, and the rays in this cone will be plane polarized. The rays of light emerging from the crystal will all be parallel to L O, that is, they will form a hollow cylinder. And if they are allowed to fall on a screen, G H, they will produce a ring of light of size independent of the distance of the screen from the crystal.

These two phenomena, called respectively external and internal conical refraction, were predicted by Sir William Hamilton as consequences of Fresnel's theory. They were discovered experimentally by Dr. Lloyd in a plate of aragonite, cut with its faces normal to the line bisecting the acute angle between the optic axes.

In the case of external conical refraction, Lloyd found that the axis of the emergent cone made an angle of 15° 25' with the normal to the plate, and had a vertical angle of 2° 59'. Calculation gave for these angles 15° 58' and 3° 1'.

In the case of internal conical refraction, the angle of incidence of the ray L O was 15° 40', and the angle of the interior cone 1° 50'. Calculation gave for these angles 15° 19' and 1° 55'.

It was found, too, that the rays, in emerging from the crystal, were plane polarized in the manner indicated by theory.

These two phenomena of conical refraction have been considered to afford a good verification of Fresnel's theory. But Professor Stokes has pointed out that other theories of double refraction would give rise to the same form of wave-surface, and lead to the same phenomena as consequences Thus they cannot decide in favour of Fresnel's particular theory.

External conical refraction depends upon there being two directions for a ray, each corresponding to an infinite number of wave-fronts, namely, the axes of single-ray velocity. These axes are sometimes called *the axes of external conical refraction.*

Internal conical refraction depends upon there being two directions of propagation of a wave-front, each corresponding to an infinite number of rays, namely, the optic axes, or axes of single-wave velocity. These axes are sometimes called *the axes of internal conical refraction.*

Let O A represent the axis of a uniaxal crystal. Along the direction O C two waves can be propagated—one with the same velocity for all positions of O C, which is said to be polarized in the plane A O C; and another with velocity depending on the position of O C, which is said to be polarized in a plane at
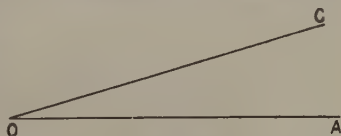


Fig. 196.

right angles to A O C. In one of these waves the vibrations are in the plane A O C, and in the other at right angles to A O C; and we have to determine in what direction are the vibrations for each wave. The vibrations at right angles to A O C are normal to the axis, and are therefore such as to give the same elasticity of the ether for all positions of O C, and the same velocity for the wave. But a displacement in the plane A O C and normal to O C gives rise to an effective component of force in its direction, which depends on the inclination of O C to the axis. Thus of the two waves travelling along O C, the one whose vibrations are at right angles to A O C travels always with the same velocity for all directions of O C, and that whose vibrations are in the plane A O C has a velocity depending on the position of O C. The first is, then, the ordinary and the other the extraordinary wave. But these are said to be polarized, respectively, in and

perpendicular to the principal plane A O C. It follows that, according to the theory of Fresnel, we must suppose the ether-vibrations in a plane polarized wave to be *at right angles to the plane of polarization.*

## CHAPTER XVII.

### *POLARIZATION BY REFLEXION AND REFRACTION. ELLIPTIC POLARIZATION.*

DOUBLE refraction is not the only method of producing plane polarized light. It was discovered by Malus that light may be plane polarized by reflexion at the surface of glass. The polarization produced in this manner is in general only partial. With a given reflecting substance the degree of polarization depends upon the angle of incidence. The angle of incidence giving the most complete polarization is called the **polarizing angle.** Brewster found that, if $i$ is the polarizing angle and $\mu$ the index of refraction of the substance, then tan $i = \mu$. The plane in which the light is polarized is the plane of incidence or the plane of reflexion. This may be proved by testing the reflected beam with a rhomb of spar or a Nicol's prism. Using a Nicol in the path of the beam, the light has a maximum of intensity when the principal plane is at right angles to the plane of reflexion, and a minimum when the principal plane coincides with the plane of reflexion. But the light admitted by the Nicol is polarized in a plane at right angles to the principal plane. Thus the reflected light is polarized in the plane of reflexion.

It appears that we may give this practical definition of the plane of polarization of a plane polarized beam : It is the plane in which a beam must be reflected so as to be polarized by reflexion and to coincide both in position and in polarization with the given beam.

Brewster's law, tan $i = \mu$, may be put into another form. For let $r$ be the angle of refraction corresponding to $i$. Then—

$$\tan i = \mu = \frac{\sin i}{\sin r};$$
$$\therefore \sin r = \cos i.$$
$$i + r = 90°.$$

But $i$ is also the angle of reflexion. Thus the reflected and refracted rays are at right angles to each other.

Jamin has shown that for most transparent substances there is no angle of incidence which will give complete plane polarization of the reflected beam, as was thought by Brewster. It is only those substances which have a refractive index of about 1·46 that can produce complete plane polarization by reflexion.

When a beam of light falls on the reflecting surface of a transparent medium, a portion will be refracted. This refracted beam is also partially plane polarized, its plane of polarization being at right angles to the plane of refraction.

**Pile of Plates.**—To obtain a beam of light polarized by reflexion, instead of using a single reflecting surface of a transparent medium, a better plan is to use a number of thin glass plates piled together, and let a beam of natural light fall on them at the polarizing angle. The component vibrations across the plane of reflexion (that is, giving light polarized in the plane of reflexion) are not completely reflected at the first surface, so that the beam of polarized light reflected from this surface alone is faint. But more of these vibrations will be reflected from the other surfaces, both front and back, of the plates, and the light falls on each surface at the polarizing angle. Thus a bright beam of reflected light polarized in the plane of reflexion is obtained.

The light refracted through the first plate is partially polarized in a plane at right angles to the plane of refraction. But on refraction through the successive plates the components of vibration in this plane are more and more completely cut out, being reflected. And thus by using several plates the transmitted light is more completely polarized.

This arrangement, called a pile of plates, is a very useful means of obtaining a beam of plane polarized light : it is generally used to polarize light by reflexion. The polarization produced by it is almost complete even for angles of incidence differing considerably from the polarizing angle.

Fresnel has given a theory of the reflexion and refraction of polarized light, of which we shall now give a sketch.

Imagine a beam of light falling at an angle of incidence $i$ on the surface of a transparent medium, and travelling with velocity $v$. Suppose it gives rise to a reflected beam, which will be reflected at an angle $i$, and will travel with velocity $v$, and a refracted beam, refracted at an angle $r$, and travelling with velocity $v'$. Now, suppose that the amplitudes of the ether-vibrations in these three beams are equal, respectively, to $a$, $b$, $c$. We shall first find a relation between these quantities

by observing that the energy of the incident beam is transferred to the other two beams.

Suppose a particle to be vibrating in simple harmonic motion under the action of a force tending to its mean position, and proportional to its displacement. The energy associated with the particle will be, as a rule, partly potential, due to its displacement, and partly kinetic, due to its velocity. The entire energy may be estimated by finding what it is when it is all potential, that is, when the particle is at the extremity of its swing, or when it is all kinetic, that is, when the particle is passing through its mean position. Now, if the period be given, the velocity in the mean position is proportional to the amplitude.

In the case of light of a given quality partly reflected and partly refracted at a surface, the vibration-periods are everywhere the same. Thus if $\rho$, $\rho'$ be the ether-densities in the two media, the energies per unit volume in the incident, reflected, and refracted beams are proportional to $\rho a^2$, $\rho b^2$, $\rho' c^2$.

But we have still to consider what volumes in the reflected and refracted beams receive the energy from a given volume of the incident.

Let the incident, reflected, and refracted beams be represented by C A D B, E A F B, A G B H. Let A C, A E, A G, be taken proportional to $v$, $v$, $v'$.
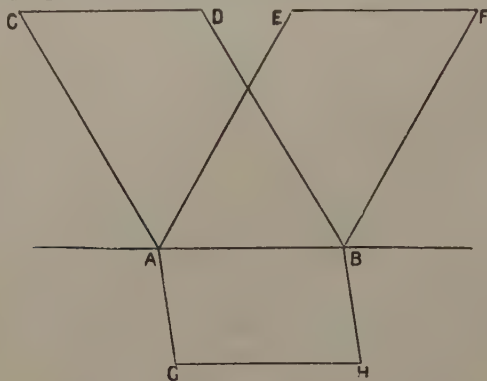


FIG. 197.

All the energy in the space C A B D will, when the disturbances at C, D have just reached A, B, be transferred to the spaces A B F E, A G H B. The breadths of these spaces, taken normally to A C, A E, A G in the figure, are A B cos $i$,

A B cos $i$, A B cos $r$. And their breadths across the figure are equal. Thus, the volumes are proportional to $v$ cos $i$, $v$ cos $i$, $v'$ cos $r$. And the equation of energy is—

$$\rho a^2 v \cos i = \rho b^2 v \cos i + \rho' c^2 v' \cos r,$$

or—

$$\rho(a^2 - b^2)v \cos i = \rho' c^2 v' \cos r.$$

And—

$$\frac{v}{v'} = \mu = \frac{\sin i}{\sin r}.$$
$$\therefore \rho(a^2 - b^2) \sin i \cos i = \rho' c^2 \sin r \cos r.$$

Now, Fresnel makes the assumption that the elasticities in the two media are the same. Thus—

$$\frac{\rho}{\rho'} = \frac{v^2}{v'^2} = \frac{\sin^2 i}{\sin^2 r}.$$

And Fresnel's energy equation is—

$$(a^2 - b^2) \sin r \cos i = c^2 \sin i \cos r,$$

or—

$$(a^2 - b^2) \tan r = c^2 \tan i.$$

Fresnel proceeds on the further hypothesis that there is no slipping between the ether particles, or that the velocities and displacements of two contiguous ones are always practically the same. Consider two such particles on opposite sides of the surface. At any instant the component of displacement parallel to the surface of the particle in the first medium, due to the incident and reflected waves together, must be equal to the same component, for the particle in the second medium is due to the refracted wave.

The still further assumption is necessary that at the surface of separation the phases of the vibrations in the three waves are always equal. Thus the maximum displacements for the three sets of vibrations occur at the same instant. But the particle on one side of the surface undergoes displacements for the incident and reflected waves, and that on the other side undergoes a displacement for the refracted wave. Thus the sum of the resolutes of $a$ and $b$ parallel to the surface must be equal to that of $c$.

We shall consider the cases of an incident beam of light polarized (1) in, (2) perpendicular to, the plane of incidence; and determine the intensities, according to Fresnel's theory, of the reflected and refracted waves in each case. A natural beam of light, incident on the surface of a transparent medium,

may be considered as having vibrations compounded of these two plane polarized beams.

**Light polarized in Plane of Incidence.**—The displacements $a$, $b$, $c$ are all parallel to the plane, so that we have here—

$$a + b = c,$$
$$\text{and } (a^2 - b^2) \tan r = c^2 \tan i;$$
$$\therefore \text{ by division } (a - b) \tan r = c \tan i.$$

$$a - b = c \frac{\tan i}{\tan r}.$$

By addition with the first we get—

$$2a = c \frac{\tan i + \tan r}{\tan r}$$
$$= c \frac{\sin (i + r)}{\sin r \cos i}.$$
$$c = 2a \frac{\cos i \sin r}{\sin (i + r)}.$$
$$b = c - a = -a \frac{\sin (i - r)}{\sin (i + r)}.$$

**Light polarized at Right Angles to Plane of Incidence.**—The resolved parts of $a$, $b$, $c$ parallel to the surface are $a \cos i$, $b \cos i$, $c \cos r$. Thus we get—

$$(a + b) \cos i = c \cos r,$$
$$\text{and } (a^2 - b^2) \tan r = c^2 \tan i;$$
$$\therefore (a - b) \sin r = c \sin i.$$

Or—

$$a + b = c \frac{\cos r}{\cos i}.$$
$$\text{and } a - b = c \frac{\sin i}{\sin r};$$
$$\therefore 2a = c \frac{\cos r \sin r + \cos i \sin i}{\cos i \sin r}.$$
$$c = 2a \frac{\cos i \sin r}{\sin (i + r) \cos (i - r)}.$$
$$b = a - c \frac{\sin i}{\sin r}$$
$$= a \left\{ 1 - \frac{2 \sin i \cos i}{\sin (i + r) \cos (i - r)} \right\}$$
$$= a \frac{\sin r \cos r - \sin i \cos i}{\sin (i + r) \cos (i - r)}$$

$$= - a \frac{\sin (i - r) \cos (i + r)}{\sin (i + r) \cos (i - r)}$$

$$= - a \frac{\tan (i - r)}{\tan (i + r)}.$$

We infer from the expression for $b$ in this case, that if the incident light is polarized at right angles to the plane of incidence, and $i + r = 90°$, there is no reflected light. Or, if the incident light is natural, and $i + r = 90°$, there is no component of vibration in the plane of incidence reflected; or the reflected light is polarized in the plane of reflexion. Thus Fresnel's theory leads to Brewster's law.

Referring to the two expressions for $c$, we see that that which refers to light polarized at right angles to the plane of incidence is always greater than the other. Thus there will always be an excess of light polarized at right angles to the plane of refraction in the transmitted beam arising from a natural incident beam.

If light, passing through the denser of two media, is incident on their surface of separation at an angle greater than the critical angle, there is no refracted ray, all the light is internally reflected. The angle $r$ is in this case imaginary. And the values of $b$, if worked out in terms of $i$ and $\mu$, will both be imaginary. Now, these values of $b$ were found on the assumption that there is no difference of phase at the surface between the incident and reflected beams. Fresnel supposed that the imaginary values indicated a difference of phase between the vibrations at the reflecting surface in the incident and reflected waves. Further, the phase-differences introduced in the two cases, where the vibrations are across and in the plane of incidence, may be different from each other. If this is so, a plane polarized beam reflected in this way will not, as a rule, give rise to a plane polarized beam; for it may be resolved into two, the vibrations in which are, respectively, in and across the plane of incidence. Thus the motion of a particle in the reflected beam will be compounded of two vibrations in directions at right angles to each other, and, in general, differing in phase; and these will not give rise to motion in a straight line.

**Elliptically Polarized Light.**—This leads us to consider another manner in which light may be polarized. Suppose a particle to have two independent simple harmonic motions along two straight lines at right angles to each other, the periods being the same, but the phases different. If $x, y$ are

the distances of the particle at any instant from its mean position, measured parallel to the two directions, we have—

$$x = a \sin \omega t,$$
$$y = b \sin (\omega t + \delta);$$

where $\delta$ is the given difference of phase-angle for the two vibrations, or $\frac{\delta}{2\pi}$ is the phase-difference, and $\frac{2\pi}{\omega}$ is the period of each vibration. From the equations we get—

$$\frac{y}{b} = \sin \omega t \cos \delta + \cos \omega t \sin \delta,$$

or—

$$\left( \frac{y}{b} - \frac{x}{a} \cos \delta \right)^2 = \left( 1 - \frac{x^2}{a^2} \right) \sin^2 \delta;$$

$$\frac{x^2}{a^2} - \frac{2xy}{ab} \cos \delta + \frac{y^2}{b^2} = \sin^2 \delta.$$

This is the equation of an ellipse having the mean position of the particle as centre. Thus this ellipse is the path described by the particle.

If $\delta$ is $\frac{\pi}{2}$, or the phase-difference is $\frac{1}{4}$, the ellipse has for equation—

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and its axes are parallel to the directions of the given component vibrations.

If $\delta$ is o or $\frac{\pi}{2}$, or the phase-difference is o or $\frac{1}{2}$, the equation becomes—

$$\frac{x}{a} - \frac{y}{b} = 0,$$

or—

$$\frac{x}{a} + \frac{y}{b} = 0.$$

Thus in each of these cases the path of the particle is a straight line.

When in a beam of light all the ether particles vibrate in ellipses whose axes have definite directions and a definite ratio, the light is said to be **elliptically polarized.** We shall see how such a beam may be produced in practice. If one of the axes of the ellipse becomes zero, the light becomes

plane polarized as a special case. If the axes of the ellipse are equal, so that the ellipse is a circle, the light is said to be **circularly polarized.**

**Fresnel's Rhomb** is a rhomb made of St. Gobain glass, which has refractive index about 1·5 ; in the figure, A B C D denotes a section. A ray of light, L M N P, falling normally on the face A B, is twice internally reflected at M and N, and emerges at P. The angles of internal incidence and reflection are made to be 55°. Then at each of the points M and N a phase-difference of $\frac{1}{8}$ is introduced between the vibrations in, and at right angles to, the plane of incidence ; and on the whole a phase-difference of $\frac{1}{4}$ is produced.

If the incident light is plane polarized in any plane, the vibrations may be considered as compounded of two sets, in and at right angles to the plane of the figure. In the emergent ray these vibrations differ in phase by $\frac{1}{4}$, and the result is elliptically polarized light. If in particular the plane of polarization of the incident light is at

FIG. 198.

45° to the plane L M N P, the emergent ray is circularly polarized. If this emergent ray is examined by means of a Nicol, it will give rise to a beam of the same intensity in all positions of the Nicol, and will in this respect resemble natural light. But if it is passed through another such rhomb in the same way, the circular vibration is decomposed into two, in and across the plane of internal incidence, having a phase-difference of $\frac{1}{4}$, and a further phase-difference of $\frac{1}{4}$ is introduced by the second rhomb ; so that the ray emerges as a plane polarized ray, the plane of polarization making an angle of 45° with the plane of incidence inside the second rhomb, as may be shown with a Nicol. Fresnel's rhomb is then a means of producing, or of detecting, when used with a Nicol, circularly polarized light.
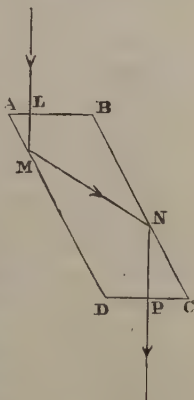
Suppose the plane of polarization of the light falling on the rhomb makes any angle (different from 0°, 45°, or 90°) with the plane of incidence in the rhomb. Let this angle be $a$. Then, according to Fresnel's theory, the direction of the vibrations makes an angle, $\frac{\pi}{2} - a$, with the plane of incidence. Let $a$ be the amplitude of a vibration. This gives rise to vibrations

of amplitudes $a \sin \alpha$ and $a \cos \alpha$ in and at right angles to the plane of incidence in the rhomb. These, on emerging from the rhomb, have a phase-difference of $\frac{1}{4}$. Thus we get elliptically polarized light. The axes of the ellipse are in and at right angles to the plane of incidence, and their ratio is $\tan \alpha$. If this light be examined by a Nicol, it will give rise to a beam of varying intensity, the maximum and minimum of intensity being obtained with the principal plane of the Nicol in the two positions parallel to the axes of the ellipse. In this respect a beam of elliptically polarized light resembles a beam of partially plane polarized light, that is, a beam consisting of a mixture of natural light and plane polarized light, such as may be obtained by reflexion at the surface of glass. But if the elliptically polarized beam be sent through another Fresnel's rhomb, so that the planes of incidence in the two shall coincide, another phase-difference of $\frac{1}{4}$ between the two component vibrations will be introduced, and the emergent beam will be plane polarized. Thus with Fresnel's rhomb we can produce, or, with a Nicol as well, detect, elliptically polarized light. If a beam of light is elliptically polarized, we may find, by trial, positions for the rhomb and the Nicol which will entirely quench the beam. Then the position of the rhomb shows the positions of the axes of the ellipse ; and the position of the Nicol with reference to the rhomb gives the ratio of the axes. For if the axes of the ellipse are $a$ and $b$, the resulting rectilinear vibration will make an angle, $\theta$, with the former of these, such that $\tan \theta = \dfrac{b}{a}$. And the principal plane of the Nicol must be parallel to the direction of this vibration to quench the beam.

Double refraction is another means that may be used for producing circular or elliptic polarization. Imagine a plate of a uniaxal doubly refracting crystal cut parallel to its axis. Suppose a beam of plane polarized monochromatic light to fall normally on this plate. A vibration is broken up into two, in and across the principal plane of the crystal which is parallel to the beam. These two vibrations will travel through the plate with different velocities. If the thickness of the plate is $e$, and $\mu_o$, $\mu_c$ are its refractive indices, the optical distances through the plate for the two vibrations are $e\mu_o$ and $e\mu_c$ ; that is, they will emerge, respectively, in the same phases as if they have traversed these distances in air. Thus the plate introduces a relative retardation between the rays of amount $e(\mu_r \sim \mu_n)$. And there will be a phase-difference between the

vibrations on emergence of amount $\dfrac{e(\mu_e \sim \mu_o)}{\lambda}$. Thus, in general, the light emerging from the plate is elliptically polarized. If, however, $e(\mu_e \sim \mu_o)$ is any multiple of $\lambda$, the vibrations emerge in the same phase, and the light is still plane polarized in the same plane as on incidence; and if $e(\mu_e \sim \mu_o) = n\lambda + \dfrac{\lambda}{2}$, the phase-difference on emergence is $\frac{1}{2}$, and the light on emergence is again plane polarized, but now in a different plane. For, let O A denote the trace of the principal plane on the plane of the figure. The vibration of amplitude O P resolves into the two O N, O M. On emergence the phase of O M is changed by $\frac{1}{2}$ relatively to O N. Thus the emergent vibrations are O N, O L, which compound into O Q,



Fig. 199.

which is equal in magnitude to O P, and such that the angles A O P, A O Q are equal. Thus the planes of polarization on incidence and emergence make equal angles with the principal plane on opposite sides of it.
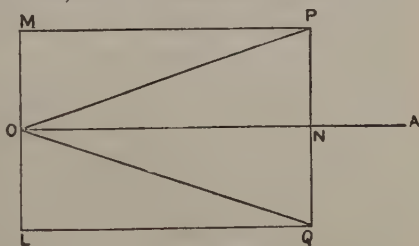
If $e(\mu_e \sim \mu_o) = \dfrac{\lambda}{4}$, the phase-difference introduced is $\frac{1}{4}$, and then we get elliptic polarization with the axes of the ellipses of vibration parallel and perpendicular to the axis of the crystal. The plate is then called a **quarter-wave plate**. Such a plate is of frequent use in physical optics; it is generally made of mica, for which $\mu_o$ is greater than $\mu_e$, or which is a negative crystal. The necessary thickness of the mica plate when used with sodium light is about 0·032 mm. It may be used, in particular, to produce circularly polarized light by passing through it a beam of light polarized in a plane at $45°$ to its optic axis.

Again, if a quarter-wave plate is interposed, normally, in the path of a circularly polarized beam, it will convert it into a plane polarized one. Or, if it be placed to receive an elliptically polarized beam with its axis parallel to an axis of the ellipse, it will convert it into a plane polarized beam. Thus with the use of the plate and a Nicol, we can find whether light is elliptically polarized, and the positions and ratio of the axes

of the ellipse. We would first find whether a position can be found for the plate such that in some position of the Nicol the light is quenched. If this is so, the light is polarized in some manner; and the position of the plate gives the positions of the axes of the ellipses of vibration; plane and circular polarization being particular cases of elliptic. Next observe the position of the Nicol relatively to the plate, when the light is quenched. This gives the ratio of the axes of the ellipse as the tangent of the angle of inclination of principal planes of plate and Nicol. If the polarization is circular, of course any position of the plate will do; any two rectangular directions normal to the beam may be regarded as the directions of the axes of the ellipses of vibration.

For use with white light a quarter-wave plate is not so good as a Fresnel's rhomb, for the phase-difference introduced by the plate varies with the quality of the light. A plate which is a quarter-wave plate for one wave-length will not be so for all others; it could only be so if the difference of indices, $\mu_e - \mu_o$, were proportional to $\lambda$. Fresnel's rhomb is better for use with white light, since it introduces nearly the same phase-difference, $\frac{1}{4}$, for light of all wave-lengths.

**Babinet's Compensator.**—A B C, B C D are two quartz wedges, having their angles at B and C equal and very small.
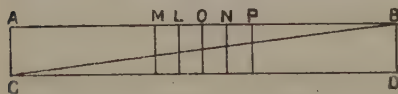


· FIG 200.

They are put together as in the figure, to form a plate. In both wedges the axes of the crystal are parallel to the faces A B, C D of the plate, but in A B C the axis is parallel to the plane of the figure, or perpendicular to the edge B; and in B C D it is at right angles to the plane of the figure, or parallel to the edge C. If a ray of light enters the compensator normally at the face A B, it is broken up into an ordinary and an extraordinary plane polarized ray in the wedge A B C. On entering the wedge B C D, these will become respectively the extraordinary and the ordinary ray, and the entire amount of relative retardation between the two rays will be proportional to the difference of the thicknesses of the two wedges through which they have passed. Let $\mu_o$, $\mu_e$ denote the ordinary and extraordinary indices of quartz, and suppose that the ray has passed through the compensator at a place where the thicknesses of the wedges are $e$ and $e'$. By the first wedge a relative retardation, $e(\mu_e - \mu_o)$, is introduced of the extraordinary

on the ordinary ray; and by the second a relative retarda-
tion, $- e'(\mu_e - \mu_o)$. Thus the entire relative retardation is
$(e - e') (\mu_e - \mu_o)$.

Jamin has employed this apparatus for the study of
elliptically polarized light. In his instrument the lower wedge
is fixed, and the other can be moved by means of a micro-
meter screw parallel to its axis. There is a fixed cross-wire
set parallel to the axis of the fixed wedge.

If a beam of plane polarized light falls normally on the
compensator, the light emerging from O, where the thicknesses
of the wedges are equal, will be plane polarized and in the
same plane as at incidence, for there is no relative retardation
between the component vibrations. Also at certain other
points, L, M, N, P, etc., where the relative retardation is a
multiple of $\lambda$, the polarization will not be changed. At the
points midway between these, the phase-difference of the
vibrations will be $\frac{1}{2}$, and the emergent light will be plane
polarized, but in a plane making the same angle on one side
of that of the figure as the plane of polarization does on the
other. Thus there will be a series of parallel and equidistant
straight lines across the compensator, such that the light
emerging from them is plane polarized, the light coming from
all the alternate ones being polarized in one plane, and that
coming from the others in a plane inclined at $2a$ to this
plane, where $a$ is the angle between the plane of polarization
of the incident light and the axis of the upper wedge.

At all intermediate points a phase-difference which is not
a multiple of $\frac{1}{2}$ will be produced; so that the light coming
from such points will be elliptically polarized.

Now, let the compensator be viewed by a Nicol set so as
to quench the light it would receive if the compensator were
removed. A series of equidistant dark bands will be seen
at the points where the phase-differences produced are
0, 1, 2, . . . The points between them will be of variable
brightness; and the greatest brightness will be obtained
between the dark bands if the plane of polarization of the
incident light makes an angle of $45°$ with the axis of each
wedge. As the upper wedge is moved by the micrometer
screw, the phase-difference of the vibrations proceeding from
any point of the field is changed by an amount proportional
to the displacement. The phase-difference introduced by a
given displacement can be found in this way: First turn the
micrometer screw till a dark band comes under the cross-wire.
Then observe the amount of motion necessary to bring the

next band under the wire. During this motion the phase-difference of the component vibrations coming from the points under the wire has gradually changed from o to I. Let the displacement be $d$. Thus a displacement $x$ will produce a phase-difference at any point of the field equal to $\frac{x}{d}$, in addition to that already existing at the given point.

To examine a beam of elliptically polarized light with the compensator, set it so as to produce no phase-difference under the wire, or, as we may say, in its zero position. Place it in the path of the beam, and observe it with a Nicol. The
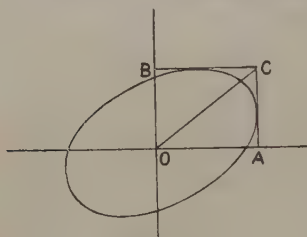


FIG. 201.

different parts of the compensator introduce different amounts of phase-difference between the components of vibration passing through it; and at certain points the phase-difference in the elliptic vibration will be just cut out, and the light reduced to plane polarized light. Thus if O A, O B denote the component vibrations, the compensator will reduce these to the rectilinear vibration O C. The direction of O C may be found by turning the analyzing Nicol till the dark bands are blackest; then its principal section is at right angles to O C.

To find the phase-difference between O A and O B in the elliptic vibration, turn the micrometer screw till the dark band nearest the wire is brought under the wire. This introduces a phase-difference under the wire which just annuls the phase-difference required; and so this is measured by the amount of the displacement.

Or, to find the position and ratio of the axes of the ellipse of vibration, we may operate as follows: If the compensator is set so that the axes of the wedges are parallel to the axes of the ellipse, the phase-difference of component vibrations on incidence is $\frac{1}{4}$. Now set the compensator so as to introduce a phase-difference of $\frac{1}{4}$ under the wire, by moving the upper wedge through a distance $\frac{d}{4}$ from the zero position. Place it in the path of the light, and turn it till, on looking at it through the analyzer, a dark band is seen under the wire. In this position, then, plane polarized light is produced, and the full phase-difference is o or $\frac{1}{2}$. The phase-difference on incidence

is $\frac{1}{4}$, and the axes of the compensator are parallel to those of the ellipse. Turn the analyzer till the bands are at their blackest; then its principal section is at right angles to the rectilinear vibrations coming from the points at which black bands are seen. Thus if the axes of the ellipse are $a$ and $b$, and the principal section makes an angle, $\theta$, with the axis $a$,

$$\tan \theta = \frac{a}{b}.$$

# CHAPTER XVIII.

## COLOURS PRODUCED BY DOUBLE REFRACTION.

### PARALLEL LIGHT.

SUPPOSE a parallel beam of monochromatic plane polarized light to fall normally on a plate of doubly refracting crystal, and to be viewed on emergence with an analyzer. To consider the effect produced.

Let O P denote the direction of the vibrations in the incident beam; O X, O Y, the directions of the vibrations in the crystal; and O A the direction of vibration in the analyzer. Thus if the polarizer and analyzer are Nicols, O P


FIG. 202

and O A will be parallel to their principal planes. Let the angles X O P, X O A be denoted by $a$ and $\beta$. Denote the incident vibration by $a \sin \omega t$. This resolves into the two vibrations along O X, O Y—
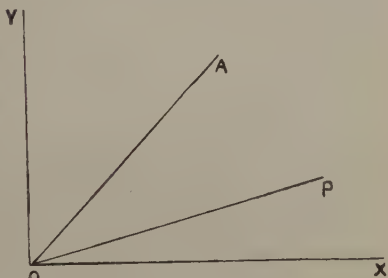
$$a \cos a \sin \omega t; \quad a \sin a \sin \omega t.$$

On emergence from the crystal these vibrations will differ in phase, on account of having travelled through the crystal with different velocities. Let the difference of phase-angle be $\delta$. Then the vibrations are—

$$a \cos a \sin \omega t; \quad a \sin a \sin (\omega t + \delta).$$

On entering the analyzer, these give rise, in the direction O A, to the vibrations—

$$a \cos a \cos \beta \sin \omega t; \quad a \sin a \sin \beta \sin (\omega t + \delta).$$

Thus the vibration along O A, which is got by simply adding these two, is—

$$a(\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos \delta) \sin \omega t + a \sin \alpha \sin \beta \sin \delta \cos \omega t.$$

This may be written—

$$A \sin (\omega t + \epsilon),$$

where $A^2 = a^2 \{ (\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos \delta)^2 + (\sin \alpha \sin \beta \sin \delta)^2 \}$,

and $\tan \epsilon = \dfrac{\sin \alpha \sin \beta \sin \delta}{\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos \delta}$

The intensity of the light transmitted through the analyzer may be denoted by $A^2$, that is—

$$a^2 \{ (\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos \delta)^2 + \sin^2 \alpha \sin^2 \beta \sin^2 \delta \},$$

or—

$$a^2 \{ \cos^2 \alpha \cos^2 \beta + \sin^2 \alpha \sin^2 \beta + 2 \cos \alpha \sin \alpha \cos \beta \sin \beta \cos \delta \},$$

or—

$$a^2 \{ (\cos \alpha \cos \beta + \sin \alpha \sin \beta)^2 - 2 \cos \alpha \sin \alpha \cos \beta \sin \beta (1 - \cos \delta) \},$$

or—

$$a^2 \left\{ \cos^2 (\alpha - \beta) - \sin 2\alpha \sin 2\beta \sin^2 \frac{\delta}{2} \right\}.$$

For light of a given colour $\delta$ has a definite value, but varies in passing from one colour to another. Thus different proportions of lights of various colours pass through the combination.

If the incident light is white (or any mixed light), the light transmitted through the analyzer may be denoted by—

$$\cos^2 (\alpha - \beta) \Sigma a^2 - \sin 2\alpha \sin 2\beta \Sigma \left( a^2 \sin^2 \frac{\delta}{2} \right).$$

The first term denotes white light, supposing the incident light white, the intensities of the constituent colours represented in it occurring in the same relative proportions as in the incident light. The second term denotes these constituent colours occurring in different relative proportions from those in the incident light. Hence the light transmitted through the analyzer will generally be coloured.

However $\alpha$ and $\beta$ be varied, the formula shows that the transmitted light is always composed of white light together with a mixture in definite relative proportions of the constituent colours; or else of white light from which such a mixture is taken away. Hence if the polarizer, or analyzer, or both, be rotated, the emergent beam always has one or other of

two definite tints which are complementary to each other. The purity of the colour of the emergent light, however, that is, the admixture with white light, varies.

If the analyzer be rotated from a given position through a right angle, the formula for the emergent light is found by writing, instead of $a$, $a + \dfrac{\pi}{2}$, and is—

$$\sin^2 (a - \beta)\Sigma a^2 + \sin 2a \sin 2\beta \Sigma \left( a^2 \sin^2 \frac{\delta}{2} \right).$$

Thus the light now obtained is complementary to that in the former position, or, if combined with it, would produce white light.

These two emergent beams may be obtained simultaneously by means of a double-image rhomb. On rotating the rhomb, the changes in the two beams are noticed; but they are always complementary; and if they partly overlap, the portion common to the two will be always white.

If $a$ or $\beta$ is equal to $0°$ or $90°$, that is, if the plane of polarization of polarizer or analyzer is parallel to one of the planes of polarization in the crystal, the second term disappears, and there is no colouring of the emergent beam.

## COLOURS PRODUCED IN CONVERGENT PLANE POLARIZED LIGHT.

We now come to consider another method by which colouring may be produced with a crystalline plate, a polarizer, and an analyzer. Suppose a beam of plane polarized light be made to converge, by means of a convex lens, to a
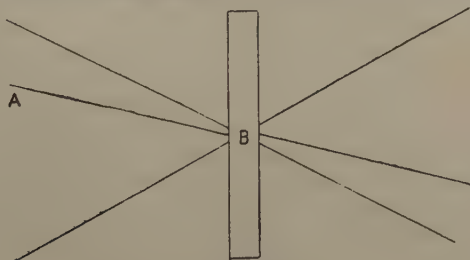


FIG. 203.

focus, and there to pass through a crystalline plate. Any ray, such as A B, will be broken up into two by double refraction; and the phase-difference introduced between the two will

depend, with a given plate, upon the position of the ray A B, that is, upon its inclination to the normal to the plate, and upon the particular plane through the normal in which it lies ; for these data determine the position of the ray with reference to the optic axis or axes of the crystal.

The light diverging from the plate is now made to converge again by a lens, or system of lenses, and passed through an analyzer. The light issuing from the analyzer in various directions will show various colours according to the phase-differences introduced by the plate.

The apparatus used for studying these effects may be designed for showing them by direct vision, the eye being turned towards the analyzer, and receiving rays coming from it along various directions, and which have passed through the plate at various inclinations. Or the apparatus may be made to show the effects by projection on a screen. Then a strong light must be used. The rays, passing through the plate at various inclinations, leave the apparatus at various inclinations, and meet the screen at different points. Thus to any point of



FIG. 204.

the screen correspond rays that have passed through the plate in a definite manner ; and a pattern is produced.
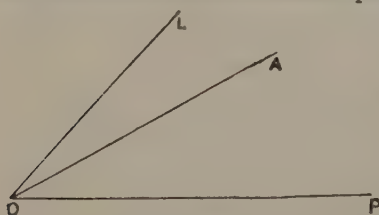
**Uniaxal Crystal cut Perpendicular to Axis.**—Let the plane of the figure be one face of the crystal. Let O P be the direction of the vibrations in the polarizer, and O A that of the vibrations in the analyzer. Let O L be the trace of the plane containing the normal to the plate and an incident ray.

If O L coincides with O P, or is perpendicular to O P, the vibrations of the incident ray are in or perpendicular to the principal plane of the crystal containing the ray, so that it is not broken up on entering. Hence the light emerging from the analyzer gives uniform illumination in the plane containing the normal and O P, or the line in the figure at right angles to O P. A cross of uniform illumination is thus produced with arms parallel and perpendicular to O P.

If O L coincides with O A, or is perpendicular to O A, the ray is not resolved by the analyzer ; that is, the analyzer produces no effect, and the ray goes on as it comes from the

plate. Thus another cross of uniform illumination is formed, with arms parallel and perpendicular to O A.

If O L has any other position, the ray will be broken up in the crystal, and a phase-difference introduced which will depend simply on the inclination of the ray to the normal, and not on its azimuth; that is, the position of the plane through the normal which contains it. Wherever the phase-difference disappears, or is a whole number, there is uniform illumination. This is the case for all the rays forming a series of right cones round the perpendicular axis. Thus in the field produced by the analyzer will be seen a series of concentric circles separated from each other by others of different illumination. The two crosses have their centres at the centre of these circles.

With white light a given difference of phase is produced at different inclinations for the different colours. Hence a series of variegated circles is seen.

These things may also be shown by means of the formula for the intensity of the light coming from the analyzer.

If the angles P O L, A O L, are $\alpha$ and $\beta$, and the phase-difference produced by the plate for any incident ray is $\delta$, the intensity of light emerging from the analyzer is, as was proved for the case of normal incidence, proportional to—

$$\cos^2 (\alpha - \beta) - \sin 2\alpha \sin 2\beta \sin^2 \frac{\delta}{2}.$$

Thus if $\alpha = 0°$ or $90°$, or $\beta = 0°$ or $90°$, we have uniform illumination proportional to $\cos^2 (\alpha - \beta)$. This shows that two crosses of equal uniform illumination are formed. This illumination will vary with the relative positions of polarizer and analyzer. If the planes of polarization of polarizer and analyzer are parallel or perpendicular, only one cross will be formed. In the first case it will be bright; in the second, black.

A line along which the colour is uniform is called an *isochromatic curve*. With monochromatic light the isochromatic curves become simply lines of uniform intensity. In the case we have just considered, the isochromatic curves are concentric circles.

The field of isochromatic curves is crossed by other regions, theoretically lines, of uniform intensity, and the colour of the incident light. These are called **brushes,** or **achromatic** lines. The crosses in the case we have considered are an example of brushes.

The determination of the form of the isochromatic lines and the brushes is, in general, a very complicated matter. We shall state the forms observed in certain other cases.

With a uniaxal crystal cut parallel to the axis, the isochromatic curves are two sets of hyperbolas conjugate to each other, with one axis parallel to the axis of the crystal.
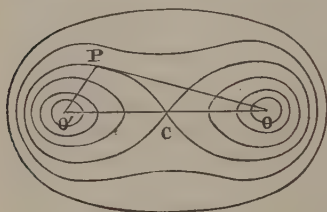


FIG. 205.

With a biaxal crystal cut perpendicular to the line bisecting the angle between the axes, the isochromatic lines are lemniscates having a common pair of foci, O, O'. Each of the curves is such that P is any point of it the product OP. O'P is constant for that curve.

The brushes are two rectangular hyperbolas passing through O, O', with the asymptotes of one parallel and perpendicular to the plane of polarization of the polarizer, and those of the other parallel and perpendicular to the plane of polarization of the analyzer. These two hyperbolas coincide if polarizer and analyzer are crossed; and are in the first case of the intensity of the incident, light; and in the second, black.

## CHAPTER XIX.

### *ROTATION OF PLANE OF POLARIZATION.*

SOME transparent substances possess the property that when plane polarized light is passed through them, it emerges plane polarized, but in a different plane from the plane of polarization at incidence. These substances are said to rotate the plane of polarization. Such a substance is quartz. If a plate of quartz is cut at right angles to the axis, it rotates the plane of polarization of plane polarized light falling normally on it. The effect is also produced by solutions of certain substances. A solution of sugar in water rotates the plane of polarization.

The plane of polarization may be rotated in either of two senses. Let us consider how these are spoken of respectively. If we look *along the direction* in which the light is travelling, then if the plane of polarization is rotated clockwise, it is said to be rotated to the right ; if counter-clock-wise,

to the left. That is, a rotation of the plane to the right is a rotation in the sense in which a right-handed screw must be turned to make it travel with the light; and a rotation to the left is one in the sense in which a left-handed screw must be turned to make it travel with the light. Some substances rotate the plane to the right, and some rotate it to the left.

In order to measure the amount of the rotation of the plane of polarization, it is necessary to have some apparatus which can be put in the path of light and turned into a definite position with reference to the plane. The thing that most obviously suggests itself for this purpose is a Nicol's prism. If this is put in the path of the light and rotated till the light is extinguished, then its principal plane coincides with that in which the incident light is polarized. If now the plane of polarization be rotated, the amount of rotation is the angle through which the Nicol must be turned from its position to again produce extinction of the light. This arrangement, however, is greatly lacking in sensitiveness; for it is impossible to tell, within several degrees, what *is* the position of the Nicol producing minimum intensity, or extinction of the light.

Other pieces of apparatus have been devised to determine with greater certainty the position of the plane of polarization. Many of these depend on the principle that it is much more easy to compare two illuminated surfaces that are seen side by side, and to determine when they are exactly alike, than to tell when a surface whose illumination varies appears darkest.

**Jellett's Prism** is made as follows: A long rhomb of Iceland spar, A C′, is taken, and its ends are cut off to form faces at right angles to its long edges. It is then cut down by a plane through the line P Q, which makes a small angle with the long diagonal, A C, of the face, and parallel to the edges A A′, etc. The surfaces thus formed are polished, one of the two pieces is turned end for end, and the two are cemented together again, as shown in the second figure. If now a beam of parallel light falls on the end of this prism normally, in each half it is broken up into an ordinary and an extraordinary beam; the ordinary passes without deviation, and the extraordinary may, by using a narrow beam and having the prism pretty long, be completely separated from the ordinary. It is the ordinary which is made use of. In the two halves of the prism the light is polarized in planes at right angles respectively to A O and C′ O. Thus the planes of polarization are inclined at a small angle.

U

If now the light passing through the prism be examined by a Nicol, the light from one half will be completely extinguished when the principal plane of the Nicol is at right angles to A O, and that from the other when it is at right angles
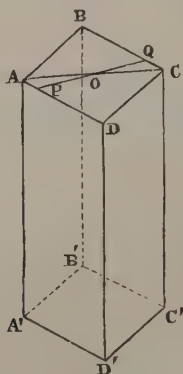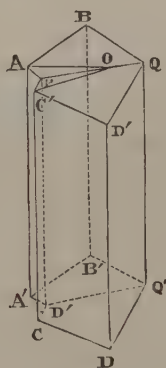


FIG. 206.                    FIG. 207.

to C′ O; since the light which passes through the Nicol is the extraordinary beam in it, and is polarized at right angles to its principal section. As the Nicol is rotated from one of these positions to the other, the intensities of illumination of the two halves of the field vary, one increasing and the other decreasing; and for a position midway between these two the two intensities are equal. This position can be fixed with great accuracy, a very small angular displacement of the analyzing Nicol from it causing the two intensities to appear very distinctly different.

There is another position of the Nicol for which the intensities are equal, namely, the position at right angles to the first. Then the intensities are both nearly at the maximum. It is the first position that should be used when both sides appear dark, for then the difference of illumination produced by a small rotation is much more readily detected.

When the position for the prism is found which makes the two parts of the field appear equally dark, if a substance producing rotation of the plane of polarization be introduced between it and the analyzing Nicol, by turning the prism into the position in which the two parts again appear equally dark, the angle through which the plane of polarization is rotated may be found.

This description shows how Jellett's prism may be used as a polarizer. It can also be used as an analyzer. For if a Nicol be used to receive a parallel beam of light, the plane polarized light produced may be transmitted to the prism ; and if now this is turned till the two halves of the field appear equally dark, the line O P must be at right angles to the principal plane of the Nicol. On introducing a rotating substance, the prism must be turned till the same appearance is again obtained, and so the amount of rotation produced is found.

**Cornu's Prism**, used for the same purposes, is a Nicol cut in two through its plane of principal section, and with the resulting faces cut away so that when the two parts are cemented together, their principal sections make an angle of about 5°. It is clear that this apparatus can be used either as a polarizer or analyzer, just as Jellett's prism.

**Laurent's Half-shade Apparatus**, which may be used with polarizer or analyzer, is made as follows : A plate of quartz or gypsum is cut parallel to the axis, and of such thickness as to produce a relative retardation of half a wavelength in sodium light. This is cut into a semicircle, and the other half of the circle is formed of a piece of glass just thick enough to absorb as much of the light as the quartz plate does.

Let O A denote the direction of the axis in the quartz plate. Suppose polarized light of the required wave-length, with its vibrations parallel to O B, to fall normally on the plate. A vibration, O B, is equivalent to two, denoted by O C and O D. On getting through the plate, the phase-difference between these two is $\frac{1}{2}$ ; that is, we have two vibrations in the same phase, denoted by O C and O E, and these compound into O F. If the plane of polarization of the incident light makes an angle, $a$, with the axis O A on one side of A, this plane is (according to Fresnel's theory) at right angles to O B, and the plane of polarization of the emergent light will make the same angle, $a$, with O A on the other side of O A. Hence if a beam



FIG. 208.

of light falls on the half-shade polarized in a plane inclined to the axis of the crystal at any angle, half of the beam will go through the glass unchanged, but the other half will have
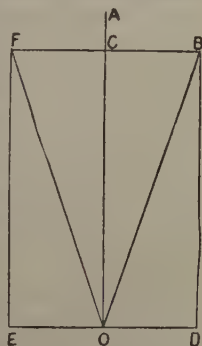
its plane of polarization rotated by the crystal through an angle twice as great as that made by the plane of polarization of the incident beam with the axis of the crystal.

It is clear that Laurent's half-shade used with a Nicol will act towards sodium light in the same manner as a Jellett's or a Cornu's prism. It is to be placed between the polarizer and analyzer. Of the beam of light which falls on it from the polarizer and passes through it, one half is polarized in one plane, being unacted upon by the half-shade, and the other half is polarized in another plane inclined at an angle to this equal to twice the angle between the axis of the crystal and the original plane of polarization. Now, when the light in either half of the beam is received by an analyzer, the intensity perceived will depend on the angle between the plane of polarization of the light and that of the analyzer, being proportional to the square of the cosine of this angle. Thus the two halves of the field, when viewed through an analyzer, will, in general, appear of different intensities. But the intensities will be the same when the plane in which the light issuing from the analyzer is polarized makes equal angles with the planes of polarization of the two parts of the beam. The half-shade is so placed that the axis of the crystal makes a small angle—a few degrees—with the plane of polarization of the light from the polarizer. Then the planes of polarization of the light coming from it make a small angle with each other, double of the above-mentioned angle. The analyzer should be set so that its plane of polarization (that is, for a Nicol, the plane at right angles to its plane of principal section) bisects the angle exterior, or supplementary, to this. Then both halves of the field will appear of small and equal illumination ; and a very small rotation of the analyzer readily shows a difference in the brightness of the two halves.

If now a substance that produces rotation of the plane of polarization be placed between the half-shade and the analyzer, the amount by which the analyzer has to be rotated to produce the same appearance again is the amount by which the plane of polarization has been rotated ; or, of course, it may differ from it by any multiple of two right angles ; but by experimenting with various thicknesses of the substance, this difficulty is removed.

Again, it is clear that the half-shade may be attached to, and rotated with, the analyzer, being placed between it and the polarizer, the axis of the crystal being set at a small angle with the plane of polarization of the analyzer (or the

perpendicular plane). The position of the plane of polarization can then be detected as before, and hence the amount by which it is rotated on placing a rotating substance between the polarizer and the half-shade.

Laurent's apparatus can only be used for sodium light (or light of any other particular quality for which the thickness of the plate may be adapted); but Jellett's or Cornu's can be used with homogeneous light of any quality whatever.

Professor Poynting has devised a very simple piece of apparatus for determining the position of the plane of polarization. A quartz plate cut at right angles to the axis is cut into two, and one half reduced a little in thickness. The two halves are then joined together again. If monochromatic plane polarized light falls on this, the plane is rotated by slightly different amounts in the two parts, and on receiving the light in an analyzer, in general, the two parts of the field appear of equal intensity. On setting the analyzer so that the lights are both nearly extinguished and of the same intensity, the plane of polarization of the analyzer bisects the large angle between the planes of polarization of the light coming from the two parts of the quartz. The apparatus may then be used like a half-shade.

The planes of polarization of lights of different qualities are rotated by any given substance by very different amounts. Hence with white light a very complex effect is produced. If various qualities are examined, it is found that the planes of polarization are rotated by amounts which are approximately inversely proportional to the squares of the wave-lengths; that is, the rotation, multiplied by the square of the wave-length, is approximately constant. M. Broch has measured the rotations produced by a plate of quartz 1 mm. in thickness on light from different parts of the spectrum. The following table exhibits the rotations produced on various rays, and the products of rotations by squares of wave-length :—

| Rays. | Rotation. | Rotation × (wave-length)². | Rays. | Rotation. | Rotation × (wave-length)². |
|-------|-----------|---------------------------|-------|-----------|---------------------------|
| B ... | 15° 18′ | ... 7238 | E ... | 27° 28′ | ... 7597 |
| C ... | 17° 14′ | ... 7430 | F ... | 32° 30′ | ... 7622 |
| D ... | 21° 40′ | ... 7511 | G ... | 42° 12′ | ... 7842 |

It appears that the product rotation × (wave length)² is not quite constant, but increases with the rotation; but to a rough degree of approximation the law above mentioned is true.

If plane polarized white light falls on a plate of quartz or of any substance producing rotation, and is then examined

with an analyzer, the intensity of the light of any given quality thus obtained is proportional to the square of the cosine of the angle between the plane in which the light of that quality is polarized on leaving the rotating substance and the plane of polarization of the analyzer. Thus the effect produced is not white light, but a colour in which the constituents of white light are mixed in different degrees. When the plane of polarization of the analyzer is at right angles to that in which light of a given quality is polarized on leaving the substance this quality of light will be completely absent from the resultant tint. It will happen, of course, that if the substance, is of sufficient thickness to rotate the planes of polarization of two qualities by angles differing by 180°, the analyzer may be set so as to extinguish both of these qualities.

If a plate of quartz of thickness not greater than 5 mms. is used when the analyzer is set to extinguish the mean yellow rays, a greyish violet tint is obtained. If the analyzer is turned a little in one sense so as to introduce a little more of the red light, the tint rapidly changes to red ; and if a little in the other sense, the tint rapidly changes to blue. The tint obtained between the red and blue is called the **tint of passage,** or the **sensitive tint.** The position of the analyzer which produces it can be fixed with considerable accuracy, for a very slight rotation from this position produces a decided change to red or to blue. Experiment shows that the position of the analyzer producing the tint of passage is the same as that which extinguishes the mean yellow rays. The readiest way to determine what rays are extinguished with a given position of the analyzer is to examine the light coming from it by means of a spectroscope. Then a dark band will be seen across the spectrum in the place of the colour which is extinguished by the analyzer.

Some specimens of quartz rotate the plane of polarization to the right, and some to the left. They are called, respectively, right-handed and left-handed. The amount by which a given thickness will rotate the plane of polarization of light of a given quality is the same, whichever way the rotation is produced.

**Bi-quartz.**—Suppose a right-handed and a left-handed plate of quartz to be placed side by side, and let plane polarized white light pass through them. Each quality of light is rotated through the same amount by the two plates, but in opposite senses. If the light is examined by a Nicol, or other analyzer, a position may be found for the analyzer in which the appearances from the two quartz plates will be the same. Either the

position in which its plane of polarization is parallel to that of the incident light, or at right angles to it, will produce this effect. For in either of these two positions light of any quality will be received from the two plates in equal intensity. Now, suppose the two quartz plates to be each of about 3·75 mms. thickness. Then each rotates the plane of the mean yellow rays about 90°; and the analyzer can be set to produce the tint of passage in both parts of the field. Such an arrangement of two quartz plates, made by shaping each into the form of a semicircle, and fastening them together to form a circle, is called a *bi-quartz.* The analyzer can be set in such a position as to give the same tint in each half with great accuracy; for a very slight rotation of the analyzer from this position causes a very perceptible difference in the two halves, one beginning to turn red and the other blue.

The bi-quartz may be used to determine the rotation produced in the mean yellow rays by any rotating substance. It and the substance are placed between the polarizer and analyzer, either of them being next the polarizer. Then in the two halves of the field the angles between the planes of polarization of the various qualities are not the same. Thus if the substance produces right-handed rotation, the light that has passed through it and the right-handed quartz will have the planes of polarization for two given qualities more widely separated than the light that has passed through the substance and the other quartz: in the first case we have the sum of the two effects; in the second, the difference. Hence it is impossible to set the analyzer in such a position as to obtain precisely the same constitution of light in the two halves of the field. But if it is set so as to extinguish the mean yellow rays, the tint of passage will be observed on both sides; and in this way the position of the plane of polarization for the mean yellow rays can be found.

The arrangement of the apparatus is shown in the figure. Light from the lamp L falls first on a polarizing Nicol, N.
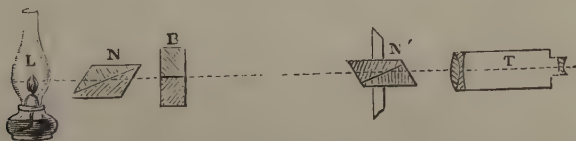


FIG. 209.

B is the bi-quartz; and N' a Nicol used as analyzer, placed at a distance from B, so that a rotating substance may be introduced

between them, and mounted on a divided circle, so that it can be rotated and the amount of rotation measured. T is a Galilean telescope, through which observations are taken. In working T should be set so as to bring the bi-quartz into focus, and to see distinctly the line between the two halves.

Another and more accurate method of working is to observe the light from the analyzer through a spectroscope. With the same arrangement of polarizer, analyzer, and bi-quartz, instead of using a telescope, a spectroscope is placed in front of the analyzer, and a convex lens between them to form a real image of the bi-quartz on the slit of the spectroscope, the line dividing the images of the two plates of quartz cutting across the slit. Two spectra will then be formed by the lights coming from the two halves of the bi-quartz, one above the other. In each of these, light of a certain quality will be wanting, according to the position of the analyzer. When the analyzer is set with its polarizing plane at right angles to the plane of polarization of light of a given quality in one of the spectra, then this light is absent from the spectrum—a corresponding dark band is seen in the spectrum. The bands in the two spectra will, as a rule, be formed in two different parts, different lights being extinguished in the two. As the analyzer is turned, these two bands approach or recede from each other; and for a certain position of the analyzer they coincide, one being formed just over the other. The light cut off from each spectrum is the mean yellow light, whose plane of polarization is rotated through 90° by each half of the bi-quartz. When a rotating substance is introduced between the bi-quartz and the analyzer, or between the bi-quartz and the polarizer, in each spectrum there will again appear a dark line, according to the position of the analyzer. But the planes of polarization of the mean yellow rays, and of these alone, coincide in the two halves, and the analyzer may be set to cut off these rays in the two halves simultaneously. On setting the analyzer, then, so that the two dark bands coincide, its plane of polarization must be at right angles to that of these rays, and so we find the amount by which the plane of polarization of these rays is rotated by the rotating substance.

**Fresnel's Theory.**—Fresnel has given a theory to account for the rotation of the plane of polarization. It depends on the fact that two circular vibrations of the same amplitude and period, but with opposite senses of rotation, are together equivalent to a linear S.H. vibration. This we shall now show.

Suppose the point P to move round the circle A B A' B' in the counter-clockwise sense, with period of revolution T (Fig. 210). And imagine the point P' to move round the same circle in the clockwise sense, with the same period. And suppose the two points to be at A together, and at A' together. Now, if we suppose a single point to be affected with both these states of motion together, its displacement from O, at the instants when P, P' are as in the figure, will be 2OM. And the point will describe a S.H.M. about O as mean position, in the straight line A A', and with amplitude 2OA.

Now, Fresnel supposes that the molecular structure of quartz, or any other rotating substance, is such that it propagates two circularly polarized beams of light, of opposite
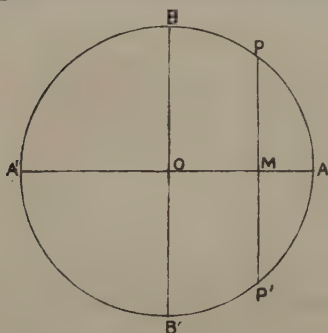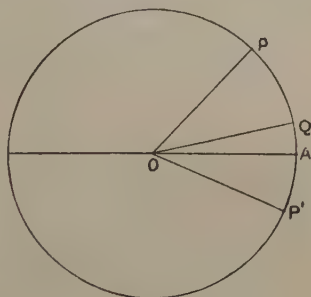


FIG. 210.    FIG. 211.

senses of rotation, with different velocities. And, further, that a plane polarized beam falling on the surface of such a medium produces the same effect as the two circularly polarized beams to which it is kinematically equivalent. Things go on, then, in the medium in just the same way as if two circularly polarized beams were passing through it with different velocities; and these, on emerging from it, compound again into a plane polarized beam.

Suppose that right-handed and left-handed circular vibrations, of period T (Fig. 211), travel through the medium with velocities $v, v'$; and have wave-lengths $\lambda, \lambda'$; so that $\lambda = v\mathrm{T}$; $\lambda' = v'\mathrm{T}'$. Let $e$ be the thickness of the medium. Let O A be the direction of a rectilinear vibration on incidence. Then this breaks up into a right-handed and a left-handed circular vibration in the medium; and these have performed by the time they emerge, respectively, $\dfrac{e}{\lambda}$ and $\dfrac{e}{\lambda'}$ complete revolutions; so that

the vibrating particles have then positions denoted by P, P ; where O P is got by rotating O A in one sense by the angle $\frac{2\pi e}{\lambda}$, and O P′ by rotating from O A in the other sense by the angle $\frac{2\pi e}{\lambda'}$. The vibrations then compound into a rectilinear vibration along O Q, which bisects the angle P O P′.

$$\text{Thus angle AOQ} = 2\pi e \left( \frac{1}{\lambda} - \frac{1}{\lambda'} \right).$$

Fresnel has further shown, by experiment, that a plane polarized beam is actually broken up into circularly polarized beams in quartz. Several prisms cut from alternately right- and left-handed quartz were joined together as in the figure.
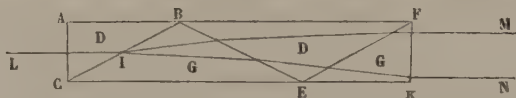


FIG. 212.

A narrow plane-polarized beam, L, was allowed to fall on them. The beam of right-handed rotation to which this gives rise will travel more quickly in alternate prisms, say first, third, etc., and the other beam more quickly in the other prisms. Thus the index of refraction for the first beam in passing from the first prism to the second is greater than unity, in passing from the second to the third it is less than unity, and so on. Hence this beam is continually deviated more and more in passing from one prism to the next. The other beam also is similarly deviated on the other side. And on emergence an appreciable separation of the two beams is produced. On examining these two beams they are found to be circularly polarized in opposite senses.

## SACCHARIMETRY.

The determination of the rotation which a substance produces in the plane of polarization has an important application in practice. It is applied to the determination of the degree of concentration of sugar in sugar solutions ; and of other substances producing rotation of the plane of polarization.

Suppose we have a substance which when dissolved in an inactive liquid (*i.e.* one not rotating the plane) produces a

rotation of the plane of polarization. The rotation of the plane produced when polarized light passes through such a solution will depend on—

(1) The nature of the substance ;
(2) The length of it which the light has to traverse ;
(3) The degree of concentration of the solution.

It seems obvious *à priori*, and it is found by experiment, that the rotation is proportional to the length employed.

Biot experimented on solutions of sugar in water, and found that the rotation produced is proportional to the mass of sugar contained in a given volume of the solution.

This law is, in general, not exact ; the rotation increasing, as a rule, not quite so rapidly as the degree of concentration, or mass of the substance per unit volume.

Now, suppose a mass, $\delta$, of a substance for which we may assume the above law, to be contained in each cubic centimeter of an inactive solvent, and let plane polarized light of a given quality traverse a length, $l$, of the solution. The rotation produced, A, is proportional to $l\delta$.

$$\text{Suppose } A = Kl\delta \quad . \quad . \quad . \quad . \quad . \quad (1)$$

K is called by Biot the **molecular rotatory power** of the substance for the given sort of light.

To determine K by an experiment, suppose $p$ grammes of the substance to be dissolved in $m$ grammes of the solvent ; and let the density of the solution be $d$. Then the volume of the solution is $\frac{m + p}{d}$ c.c. Therefore the mass of the substance contained in a cubic centimetre is—

$$p \div \frac{m + p}{d} = \frac{pd}{m + p}.$$

Thus we have the formula—

$$A = \frac{Klpd}{m + p} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The value of K is affected by the temperature ; for some substances it increases and for some it diminishes with rise of temperature.

From the above formula (1) it is clear that if for any substance we know the value of K, and we can determine the rotation, A, produced by a known length, $l$, of a solution of it, we can determine the concentration, $\delta$, of the solution. An apparatus for making this determination, or by means of

which we can determine K with the help of formula (2), is called a **saccharimeter** or **saccharometer.** It has already been indicated how apparatus can be arranged to determine the rotation which a given substance produces in the plane of polarization; and any such apparatus may be employed as a saccharimeter.

**Laurent's Saccharimeter,** of which the half-shade forms a part, is illustrated in the accompanying figure. At A is a
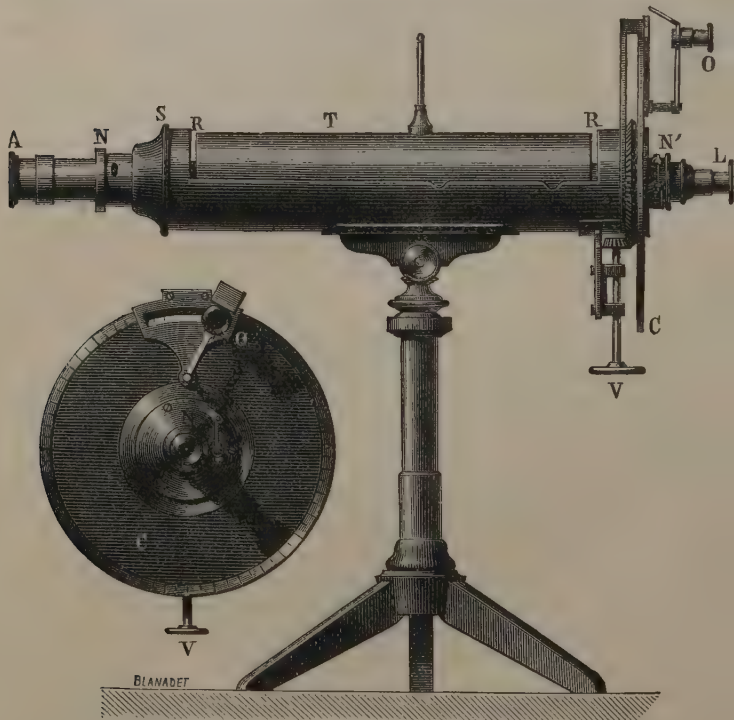


Fig. 213.

plate of bichromate of potash, to absorb all the light but the yellow rays coming from the source, which is sodium light. At N is the polarizing Nicol; at S, the half-shade. R R is a tube to contain the solution. At N' is the analyzing Nicol, which is turned with its divided circle C; and for a slow motion, to make the final adjustment, the bevelled wheels can

be brought into gear, and the analyzer turned by means of a milled head, V. The position of the circle is read by means of a vernier and a lens O. L is a small Galilean telescope, with which the half-shade is focussed.

The position of the half-shade is not fixed with reference to N; but the angle which the axis of the crystal makes with the polarizing plane of N can be varied a little. When this angle is made very small, both sides appear very dark at the same time; and if it passes below a certain limit, the intensities are too small for accurate adjustment. The setting which gives the most consistent readings is to be determined by experiment.

In making a determination of the position of the plane of polarization, the Nicol should be set in each of the two positions, which are 180° apart, giving the two halves of the field of very faint and of equal illumination, and both readings taken. In this way small errors in the construction of the instrument are avoided.

**Soleil's Saccharimeter** has the arrangement of its parts as shown in the figure. N′ is a polarizing Nicol; Q′, a bi-
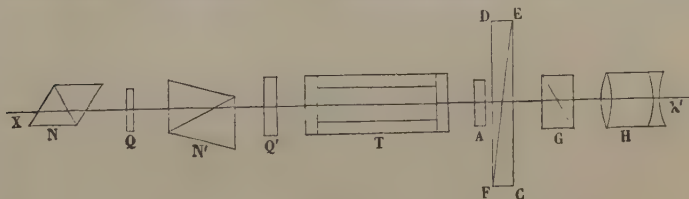


FIG. 214.

quartz; T, a tube to contain the solution; A is a right-handed quartz plate cut at right angles to the axis; D E F, E F C are two wedges of left-handed quartz cut so that the plate formed by them is also at right angles to the axis; these are movable by means of a double rackwork, so that any desired thickness can be interposed in the path of the light. G is an analyzing prism, and H a Galilean telescope which can be focussed on Q′. A scale fitted to the quartz wedges and rackwork shows the difference between the thickness offered by them and that of the plate A; it reads zero when these thicknesses are equal; or it may be graduated so as to show directly the concentration of the substance for whose analysis the apparatus is used. Or again, the zero of the scale may be in quite another position, and then the readings for equal thicknesses must be found by experiment.

The polarizing planes of N' and G being coincident; when the wedges are set at the zero reading, they with A produce no effect, and the tint of passage is seen on both sides of the bi-quartz. When a rotating substance is introduced in T, the planes of polarization for the mean yellow rays coming from Q' will be rotated, and to neutralize this effect a greater or less thickness of the wedges than that of A must be used, according as the substance is right-handed or left-handed. Then by means of the scale we can determine the thickness of quartz corresponding to the length in T of the substance used, or the degree of concentration at once, according to the graduations of the scale. N and Q are a Nicol and a quartz plate to be used if the liquid in T is coloured. By their means the light emerging from N' is made to be complementary in colour to the liquid, and then the appearance is as if the liquid were colourless.

## CHAPTER XX.

### *COLOUR.*

We have seen how, by means of an arrangement of prisms, white light may be decomposed into its constituent parts, producing colours. Suppose that a pure spectrum has been produced on a white screen, so that each point in the spectrum is illuminated by light of a definite refrangibility. With a given material for the prisms, the colour at any point of the spectrum will be determined if the refrangibility of the light illuminating that point is given. The colours at the various points of the pure spectrum are called **simple,** or **spectral,** colours, and the light corresponding to a simple colour, that is, light of a definite refrangibility for a given refracting material, is called **monochromatic** light. By combining two or more of the spectral colours, another colour may be formed. For example, some spectral colours may be reproduced by suitably combining two on opposite sides in the spectrum of the colour to be imitated; and the result may be made identically the same as the simple colour so far as the effect conveyed through the eye is concerned. But physically the light in the simple colour and that in the combination are very different; for the latter, if passed through a prism, would be broken up again into its constituent parts, whereas the simple colour cannot be further broken up.

Let us next consider what are the characteristics of a colour. These are three in number.

The **hue** of a colour is determined by the spectral colours and the quantities of them in its composition. As, for instance, the spectral colours themselves all differ from each other in hue.

The **luminosity**, or **brightness**, is the light given off by the coloured surface compared with that given off by a standard luminous surface.

The **purity** denotes the degree of freedom of the colour from admixture with white light.

It should be noticed that each of these qualities may depend on each one of two things—the nature of the coloured surface and the nature of the light used to illuminate it. Thus we have different colours of various objects all illuminated by white light, and different colours of the various parts of a white screen on which a spectrum is thrown.

Maxwell has explained the characteristics of colour, using different names from those just given ; thus : Taking the case of two lilacs, he says, " In the first place, one may be *lighter* or *darker* than the other ; that is, the tints may differ in **shade**. Secondly, one may be more *blue* or more *red* than the other ; that is, they may differ in **hue**. Thirdly, one may be more or less decided in its colour ; it may vary from purity on the one hand, to neutrality on the other : this is sometimes expressed by saying that they differ in **tint**."

**Tone** has been used as equivalent to *hue ;* and **fulness,** or **saturation,** for *purity*.

Maxwell's apparatus for mixing and matching colours, generally known as Maxwell's colour-top, consisted of two sets of coloured cardboard discs of various colours, one of the sets being of a larger, the other of a smaller diameter. Each disc was pierced with a central hole, so that any of them could be set on a circular plate with a vertical axis and spun round. Further, each disc was slit along a radius, so that any number of them could be combined, each one showing any desired sector.

Suppose a number of the discs to be combined in this way and spun about the axis quickly. Then the appearance that will be presented is that of a uniformly coloured circle, because of the persistence of impressions of the retina, and the colour obtained will be the mixture, in the given proportions, of the colours of the disc used. As an example of the use of the top, large discs of vermilion, emerald green, and ultramarine were used to form a match with two small discs of white and black. The large discs are put on first, and the small ones

above them, so that both the combinations may be seen at once. The quantity of each colour used is given by the angle of the corresponding sector. The whole circumference is divided into 100 equal parts, and this quantity was estimated in terms of one of these parts. In the example quoted, a match was made by using the following quantities of the colours : vermilion, 37 ; ultramarine, 27 ; emerald green, 36 : white, 28 ; black, 72. The result of the experiment may be expressed in the form of an equation, thus—

$$37V + 27U + 36EG = 28W + 72Bk.$$

This is called a *colour-equation.*

It appears that the three colours used, when combined in the proportions denoted, produce white. The reason why the white with which the combination is matched has to be largely diluted with black is because of the small luminosity of the combination compared with the surface of the white disc. The combination produced is in reality a neutral grey. The same three colours could also be combined in such proportions as to produce some decided tint. By suitably adjusting the relative proportions of the colours, we should obtain the right hue ; by adding white to them, we should alter the purity ; and by adding black, we should alter the luminosity. To match a given colour with the three, it would, as a rule, be necessary to add both white and black to them for the reasons given.

It will be well to notice carefully what is the exact significance of such an equation as that written above. It expresses nothing whatever with regard to the physical nature of the light. In fact, the two sorts of light which are represented as equal would be found on analysis by prisms to be quite different. The significance of the equation is purely physiological. It denotes that the two combinations of light produce the same effect on the eye of the observer who has obtained the equation, or for whom it is true. Maxwell, by comparing colour-equations got by various observers, was led to infer that the apparatus of vision, so far as the sensations of colour are concerned, differ very little in different people, excepting those people in which some part of it is entirely wanting, that is, colour-blind people. Greater discrepancies arise in the equations through using light of different qualities (daylight, gaslight, etc.) than through their being obtained by various observers.

Another piece of apparatus used by Maxwell is what is known as his colour-box. It is an improvement on the top in that with it pure spectral colours are employed instead of those reflected from coloured paper.

The box, shown in the figure, is about five feet long, and blackened inside. Light from the aperture E, passing through the lens L and the prisms P, P′, would give a pure spectrum at X Y Z. Hence white light entering by apertures at X Y Z
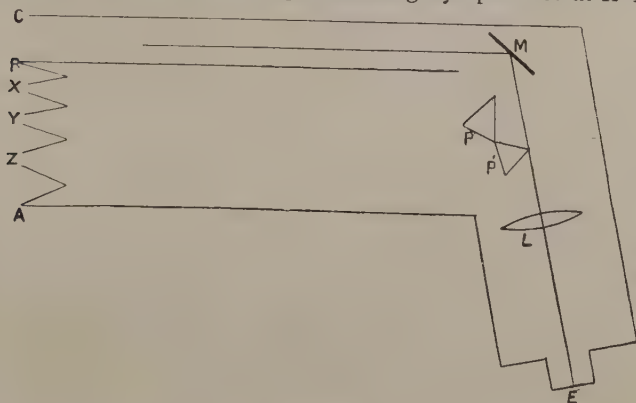


FIG. 215.

will give the same colours at E as would be produced at these apertures by white light proceeding from E. At X Y Z are three slits which can be adjusted in position and width, the widths being measured by a tapering prism. To use the box, the end A C is turned towards a white illuminated surface, so that an eye at E perceives a mixture of colours depending on the slits X, Y, Z. At M is a plane mirror which reflects some white light through the lens to E for the purpose of comparison with the mixture from X Y Z; the two portions of the field seen by the eye being separated by the edge of the prism P′. By suitably adjusting the slits, these two portions may be made identical, so that the edge of the prism becomes almost invisible.

Captain Abney has used apparatus for colour experiments which is further improved. He took care to employ an invariable source of white light, and used for the purpose the light from the crater of the positive pole of the electric arc lamp. With this he used an apparatus which he calls a *colour-patch apparatus*. An image of the crater is formed on the slit $S_1$ of a collimator. The light then passes through two prisms, $P_1 P_2$, and a lens, $L_3$, and would give a pure spectrum on a screen, D, which is slightly inclined because the violet focus is rather nearer to $L_3$ than the red. If a narrow slit is

X

used in the screen at D in the position corresponding to any simple colour of the spectrum formed, then by means of a suitably placed lens, $L_4$, an image of this colour of the near
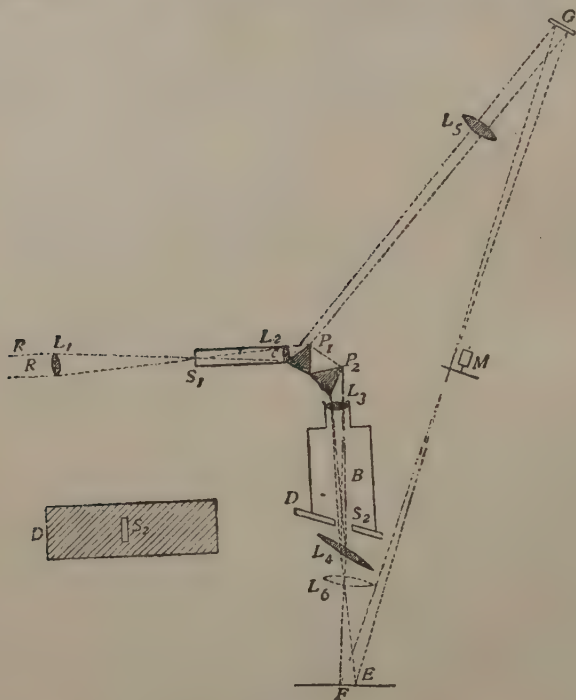


FIG. 216.

surface of $P_2$ can be formed on a screen at F. $L_4$ must be placed slightly askew, because of the differences in its focal lengths for lights of the various colours. If several slits in the screen at D be·used at once, the corresponding images will be accurately superposed on the screen at F.

To obtain a patch of white light, the light from the collimator which is reflected at the first surface of $P_1$ is used. By means of a lens, $L_5$, and a mirror, G, this is concentrated on the screen at F, where it may be superposed on or thrown alongside of the other patch. It may also be made of the same size as the other.

To adjust the luminosity of the white patch, another piece

of apparatus is used, which is represented at M. This consists of a circular disc placed in the path of the light. It can be rotated at a high speed, and has two opposite sectors cut away, which can be more or less filled up at will by a suitable arrangement during the rotation. $L_6$ is a lens of short focal length, which may be used when it is desired to see what colours are being used, for it separates the images of the various colours, forming them side by side on the screen.

**To compare the Luminosities of Lights from Different Parts of the Spectrum by Means of the Colour-Patch Apparatus.**—Let the slit $S_2$ be placed in any position in the spectrum so that the screen is illuminated by light of the corresponding colour. The rotating sectors are in position at M. A rod is placed before the screen F E, so that two adjacent parts of the screen are illuminated, one only by the coloured, the other only by the white light, the rod being used as in Rumford's photometer. The rotating sectors have then to be adjusted till the two patches are of equal brilliancy. It is not so easy a matter to decide this when the lights are of different qualities ; but Abney gives the following device : Set the sectors at a definite opening. A position for the slit $S_2$ can be found where the coloured patch is clearly too dark, and by moving $S_2$ another colour for the patch can be found which is clearly too bright. The limits between too dark and too bright can be narrowed till a pretty accurate position is found for $S_2$. The luminosity of this part of the spectrum is proportional to the aperture of the sectors. In general, two positions for $S_2$ will be found corresponding to a given luminosity, one on each side of the brightest part.

In this way the relative luminosities of all parts of the spectrum for a given observer may be found.

**The Colours of Objects** are due to their surfaces absorbing lights of some sorts, that is, of some refrangibilities more than others, and reflecting more of the other sorts. An analysis of the colour of a given pigment, for example, can be made by means of the colour-patch apparatus. A circular disc of card is painted



Fig. 217.

with the pigment, and two larger discs of black and white, which can be set to show overlapping sectors, as in Maxwell's
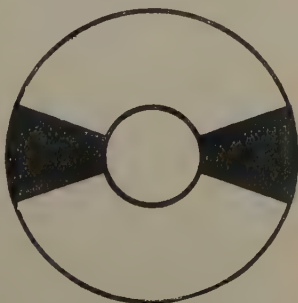
top, are set on an axis about which they can be rapidly rotated. Let the colour-patch fall partly on the inner disc and partly on the outer ring. Let the white be set to show a given extent of sector, and, on rotating the apparatus, move the slit till the brightness on the inner disc and on the outer ring is the same. The setting of the sectors will show what fraction the light of this particular sort reflected by the disc is of that which would be reflected from a white surface. In an experiment of this sort a correction has to be made, for it is found that the black surface reflects some light. The ratio of its luminosity to that of a white surface can be found by the help of the rotating sectors, and the necessary correction made. By altering the amount of white sector exposed, and finding in every case the corresponding colour, we can draw up a table, and hence plot a curve showing the intensities of light of the various sorts reflected from the surface.

To analyze the light transmitted through a given thickness of a coloured body, the colour-patch apparatus has been used in the following manner : The light from the collimator is divided into two beams by a doubly refracting Iceland spar prism. This gives two spectra, and a long slit is used at $S_2$ to go across them both. The light of any colour from one spectrum produces a patch on the screen as usual. That from the other is turned aside by a reflecting prism, and by another is reflected through a lens to the screen, giving another patch superposed on the first. The rod is used to give two regions illuminated respectively by these two lights, and the rotating sectors are set in the path of the first to equalize the intensities. The intensities will then be equal for lights of all colours. The substance is now interposed in the path of the second beam of light, and the sectors adjusted till the intensities are again equalized. We can thus compare the light of any colour that gets through the substance with the entire quantity of light of the same colour that falls upon it.

Some of the most important experiments done with the colour-patch apparatus are on the matching of colours. Three slits are placed in the spectrum, and certain positions can be selected for them, so that, by suitably adjusting their apertures, any colour whatever, whether simple or not, can be matched by the light coming through them. The colours corresponding to the positions so chosen cannot themselves be matched by any other colours, and are called the **primary colours**. They are in the *red, green,* and *violet* parts of the spectrum. The primary colours can, in the first place, be used to make

white light. The proportions of the colours used will, of course, depend on the quality of white light that has to be matched. It is found that any colour between the red and the green, if some white is added to it, can be matched by mixing the red and the green; and any colour between the green and the violet, with a certain quantity of white added, can be matched by a mixture of green and violet. From these facts we deduce an important conclusion. Suppose that any given colour is matched by a certain combination of the red, green, and violet. Then, in whatever proportions these colours occur in the combination, this may be represented as a certain quantity of white light, together with a residue of two of the colours, one of the two being green; or else as a certain quantity of white light with a deficit of two colours, one of which is green. And since green with red or green with violet gives a simple colour mixed with white, it follows that any colour whatever is equivalent to white to which a simple colour has been added or from which a simple colour has been taken away. We can thus refer the hue of any colour whatever to a spectral colour. If the hue is a combination of red and violet, or a purple, no such hue exists in the spectrum; but we can find a spectral colour which with it will make white. Now, the colour which when added to a given colour will make white is called the **complementary** colour to the given one. In the case, then, of a purple, its hue must be referred to its complementary in the spectrum.

The effect of mixing lights of given colours is quite different from the effect of mixing the pigments which have the same colours. The colour of a pigment is due to the absorption by it of light of certain colours. The colour of a mixture of pigments, then, is the colour of the light which remains from white light when the pigments have absorbed all the light of various sorts which each of them separately can absorb.

In all that has been said about matching colours, it is understood, of course, that the match is only made so far as action on the eye is concerned. When, for example, the three primary colours are combined to produce white light, the physical nature of the light produced is different from that of the light of the recombined spectrum, only it affects the eye in the same way. The explanation of this is physiological. It leads to a theory concerning the apparatus of the eye for perceiving colour which was given by Thomas Young and supported by Helmholtz, and is called the Young-Helmholtz theory. It is supposed that the eye is furnished with three

sets of colour-perceiving nerves, each set being affected most strongly by light of some given colour, but also by other colours as well, to amounts gradually falling off as we pass away in the spectrum from the colour producing the greatest effect on that set of nerves. White light, of a given quality, affects all the nerves, each set to a definite extent. A certain combination of the three primary colours affects the nerves in just the same way, and hence produces the same effect on the eye. It should be noticed in such a case that *each set* of the nerves is similarly affected by the colours that match.

Now, white light may be produced, not only by the primary colours, but with the slits occupying other positions in the spectrum; in fact, an indefinite number of corresponding positions may be found for the slits, the apertures each time being suitably arranged. In this way, from the identity of effect produced by two different combinations, colour-equations can be formed, each one of which tells us something about *each set* of the nerves. From such equations curves have been drawn showing the degree of excitation for each set of nerves caused by each colour.

Such a set of curves is shown in the diagram. From these it should be clear why certain spectral colours cannot be



VIOLET    BLUE    GREEN  YELLOW ORANGE   RED
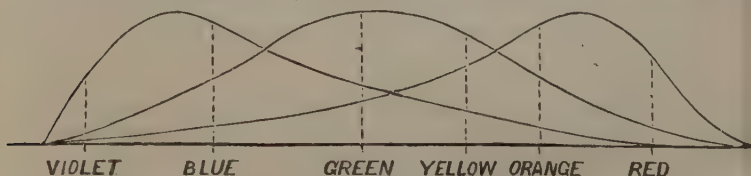
FIG. 218.

reproduced by combination of others. For, if we take the colour giving the greatest ratio of excitation of green to that of red, for example, no combination of other colours could reproduce this ratio.

Some people are found to be deficient in the colour-perceiving apparatus, lacking one set of the nerves—the absence of the red is the most common. Such people are called colour-blind. Colour-matches can be formed for such people by two simple colours, and corresponding colour-equations can be deduced. These have led to curves of excitation for the nerves which they possess, which curves agree with those traced for the corresponding nerves in normal-eyed people, and so confirm the theory in a remarkable manner.

## CHAPTER XXI.

### *SPECTRUM ANALYSIS.*

WHEN white light passes through a prism it is broken up
into its constituents of various qualities or refrangibilities.
To further separate the constituent rays from each other, the
light may be passed through a series or *train* of prisms. The
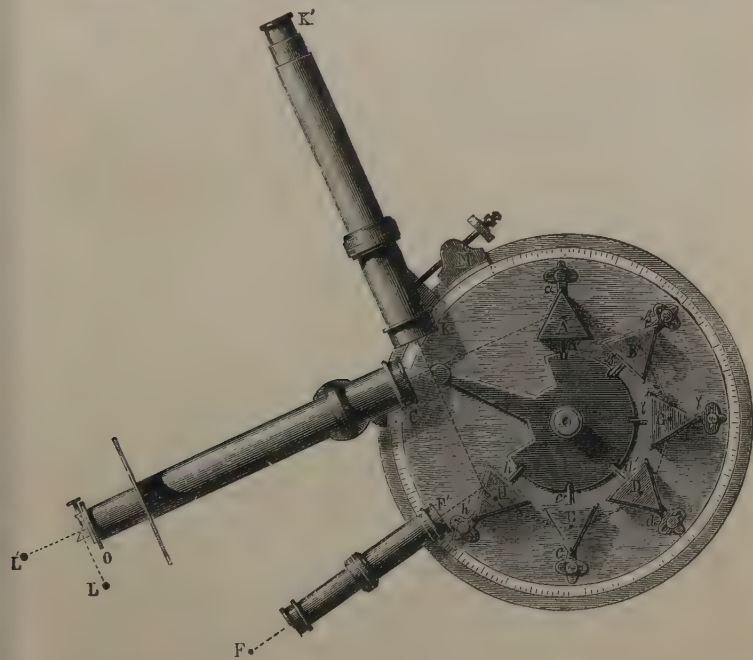


FIG. 219.

apparatus with which this is done is called a **spectroscope.**
On a platform furnished with a divided circle is a series of
prisms, A, B, etc.  O C is a collimator, having a slit at O
similar to that in the spectrometer. The light from a source,
L′, can be sent through the slit into the collimator; or, by
means of a reflecting prism, the light from a source, L, at the
side can be used. The prism only covers half of the slit, so
that both sources can be observed simultaneously. The light,

after passing through the prisms, enters a telescope, K K′, movable by a tangent screw, M. In using the instrument, each prism should be set so as to produce minimum deviation of the light in the particular part of the spectrum under examination. F F′ is a second collimator having a fine scale on glass in its focus. An image of this can be seen through the telescope by reflexion at a face of the prism, H ; so that the parts of the spectrum seen can be referred to this scale.

A **Direct-vision Spectroscope** is a convenient apparatus to use for some purposes. By its means, dispersion of the rays may be produced without any considerable deviation. We have seen that a flint and a crown glass prism may be combined so as to produce dispersion of the rays, but no deviation of a selected one. By combining several such prisms a considerable amount of dispersion may be produced, while the rays of some particular colour, say the mean yellow rays, are not deviated. The figure shows a combination of three
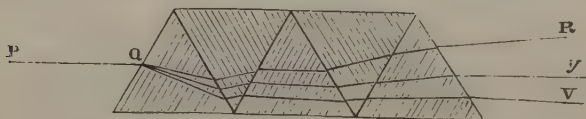


Fig. 220.

crown and two flint prisms; and shows how the ray P Q is broken up into rays going to R, *y*, V. The light, before falling on the first prism, passes through a narrow slit parallel to the refracting edges. On the other side of the prisms is a convex lens. This may be focussed on the slit ; so that light of any given quality leaves this lens in parallel rays, and produces an infinitely distant virtual image of the slit. Thus a spectrum is formed suitable to be observed by applying the eye to the lens. By moving the lens further away from the prisms, a real image of the slit for light of any given colour would be formed. Thus a spectrum may be thrown on a screen.

The spectroscope may be applied to examine the light emitted by a luminous body. In the case of a solid body, if it is gradually heated up from a low temperature at which it is not luminous, it will in time begin to emit light of the least refrangibility, *i.e.* red light. As the temperature is continually raised, the body continually emits, in addition, light of higher and higher refrangibilities ; so that, at a very high temperature, the light from it fills the whole of the visible spectrum. This could be illustrated by placing a wire in

front of the slit and passing a gradually increasing electric current through it, so as to gradually raise its temperature. We should see its spectrum begin with red light alone, and gradually spread to the blue. And as light of higher refrangibility begins to be emitted, that which is already in the spectrum becomes more intense. From a liquid body, too, which can be raised to a very high temperature, such as molten metal, we should get the same sort of continuous spectrum.

In the case of an incandescent gas, however, or of any body in a state of volatilization, by whatever means this state may be produced, the spectrum is quite different. Such a substance does not emit light of all refrangibilities, but only of some definite refrangibilities. If, for instance, a salt of the metal sodium, such as common salt, be volatilized, as can be done by burning a spirit-lamp with a salted wick, or by putting common salt (chloride of sodium) into the flame of a Bunsen burner, the salt will be decomposed, and sodium vapour will be in the flame. This will be seen to glow with a bright yellow light, and if it is examined by means of the spectroscope, the spectrum will be found to consist of a bright yellow line; or, if the spectroscope has great decomposing power, of two yellow lines, quite close to each other— a *double line* it is called. At a very high temperature the spectrum contains, in addition, other double lines. Again, if a piece of copper is held in the flame, the flame becomes green, owing to the presence of volatilized copper, and the spectrum consists of green bands. An incandescent solid (or liquid) mass of copper gives, on the contrary, the ordinary continuous spectrum.

A body may be volatilized by other means for the purpose of producing its spectrum. A small piece may be put between the poles of an electric arc lamp, and subjected to the high temperature of the arc. To obtain the spectrum of a metal, it may be made into terminals, between which a series of electric sparks is passed. This produces volatilization of the terminals, and the spark will show the characteristic spectrum of the metal. The sparks may be passed by means of an induction-coil; and a better result is obtained if a condenser is placed in the secondary circuit of the coil so as to increase the quantity of electricity passing at each discharge. In a spectrum obtained in this way, however, other lines besides those characteristic of the metal used will be observed. The spectrum of the air through which the spark is passed will present itself. This will be most clearly seen in the part of the spark farthest away from the terminals, that is, about

midway between them. The air spectrum has been obtained by Dr. Huggins by observing the spectra got with gold terminals and with platinum terminals, and taking the lines common to both as those due to air.

The spectrum of a gas may be observed by enclosing it at very small pressure in a tube furnished with terminal electrodes for the passage of a spark (Geissler's tube). A portion of the tube is made of capillary bore to increase the brightness. If a series of discharges from an induction-coil is passed through the tube, the temperature of the gas is raised to such a height that it becomes luminous, glowing with a characteristic colour, and the spectrum of it will be found, in general, to consist of lines of light.

In all cases in which a line in the spectrum is observed, this line is an image of the slit formed of light of one definite refrangibility. It will consequently become narrower as the slit is made so. And in the case of an absolutely pure spectrum, which would require an indefinitely narrow slit, such a line would be merely a geometrical line, having no width. It should be denoted as such in a representation or map of the spectrum, and in practice it can be made as fine as may be desired by narrowing the slit.

A source of light may vary in constitution in its various parts, as in the case already mentioned of the electric spark in air. The various parts may be examined by admitting the light from them in turn to the slit of the spectroscope. But the following more convenient method has been used by Lockyer : The source is placed at some distance from the slit, and by means of a lens an image of it is formed on the slit. The light proceeding from the various points of the slit is then that proceeding from the various points of some line of the source. And if this varies in quality, the spectrum observed will not be uniform throughout its breadth ; but each elementary strip of it, taken in the direction of its length, will correspond to some point of the source. Thus if the composition of the source varies continuously, we may have certain lines becoming fainter and disappearing as we pass across the various longitudinal strips of the spectrum. The cross-section of the electric arc, taken midway between the poles in which salts of metals are being volatilized, may be examined in this way. The result is a spectrum of a series of bright lines, all of which cross the centre, but which are of various lengths. Thus we have the phenomenon of *long lines* and *short lines* in the spectrum.

The spectrum obtained from a chemical compound has different lines from those of its constituents, except when a high enough temperature is used to decompose the compound. Thus if chloride of calcium is put into the Bunsen flame, the spectrum produced is that of the salt; but if the salt is put on one of the electrodes through which a powerful discharge is passed, the spectrum of calcium will be obtained. We have already seen that, by putting a sodium salt into the Bunsen flame, the sodium spectrum is obtained, since the salt is easily decomposed. In the air spectrum, if much moisture is present, the hydrogen lines are seen, since the water is decomposed by the spark. In the spectrum of lightning, the sodium line has been observed, due to the decomposition of the sodium chloride in the air.

There are definite spectral lines characteristic of a given substance, but a given substance does not by any means always present the same spectrum. Thus when we have a certain spectrum of lines, continually raising the temperature will frequently cause other lines to appear elsewhere in the spectrum. This has already been mentioned for sodium. Again, in addition to the line spectrum, a substance will present, as a rule, a spectrum of *fluted bands*, the band spectrum being got at a lower temperature. The bands appear, as a rule, to have a well-defined edge on one side, and gradually to fade away on the other. But on examining them with a spectroscope of great dispersive power, each band is seen to consist of a number of lines which gradually get closer together as the sharp edge is approached.

The lines characteristic of a given substance are not always lines of definite refrangibilities. Certain circumstances may cause these to widen out. In general, it is increase of pressure which produces this effect, the spectrum becoming more continuous as the density of the substance is increased. Increase of intensity of the electrical discharge producing the spectrum, however, will also have the effect of widening the lines.

**Absorption Spectra.**—We have next to consider the way in which certain qualities of light are absorbed by certain bodies. An incandescent vapour which gives off vibration of a definite sort also readily absorbs that particular sort of vibration. This may be illustrated by the following experiment: Suppose a continuous spectrum from a very bright source to be obtained; and then place between the source and the slit a Bunsen flame or spirit-lamp flame containing sodium vapour. This vapour would of itself give in the spectrum a bright

yellow tint; but in the present case a line will appear in the place corresponding to the sodium line, but dark as compared with what surrounds it. In the same way, if white light from a source of high temperature is allowed to pass through any vapour at a lower temperature, and to form a spectrum, we obtain a **reversal of the lines** characteristic of the vapour, that is, dark lines in the place of the usual bright ones.

Stokes has explained this absorption in this way: A molecule of a vapour has a definite vibration-frequency, or definite vibration-frequencies natural to itself. It can thus emit light of definite quality or qualities, and no other. If vibrations in the ether of such a quality fall on it, they can set it in motion, and it will absorb them; but if vibrations of a different quality fall on it, they will not influence it, and will not be absorbed by it. Just in the same way, a stretched string will emit a definite note, that is, of a definite vibration-frequency; and if this note is sounded near it, that is, if vibration of this frequency is produced in the air, the string is set in motion, but a different note will not, as a rule, affect the string.

Attempts have been made to explain, on mechanical grounds, some other facts connected with spectra. Thus to explain why a gas should give a line spectrum, but a solid body a continuous one: the gas molecules have definite frequencies with which they vibrate. They are seldom in contact with each other, and their collisions do not much influence their rates of vibration. When the molecules are crowded together, however, their natural vibrations are interfered with, and the vibrations which they emit into space are of all frequencies. Again, starting with a gas in a state of great tenuity, it gives a line spectrum, the molecules all vibrating, practically always, in their natural periods. But as the pressure increases, that is, as the molecules are more crowded together, the collisions become more frequent, and the vibrations of the molecules more and more affected by them. Or, on greatly increasing the temperature, the velocities of the molecules increase, and so does the frequency of the collisions. In either case, then, the vibrations emitted cease to be of so definite a character; the quality of the light is more complicated, and the lines widen.

Bunsen and Kirchhoff, who were among the first to study this subject with care, showed, in 1855, how the spectra emitted by substances could be used for their detection; and pointed out what extremely minute quantities could be detected with certainty by this method. All the alkaline metals, for example,

give characteristic spectra which are easily obtainable from salts of them, and which are very useful in detecting their presence. This method of determining the presence of a substance is called **spectrum analysis**.

One of the chief results to which spectrum analysis has led has been the investigation of the constitution of the atmospheres of the sun and other heavenly bodies. The pure spectrum obtained from the sun is not continuous, but is crossed by a multitude of dark lines. These lines were closely observed by Fraunhofer, and are called Fraunhofer's lines. These lines have been denoted by letters of the alphabet, beginning with A in the red, by which they are generally known. Thus there is a double line called the D line in the yellow, exactly coinciding with the double line given by sodium vapour. They are due to absorption. The radiation coming from the intensely hot body of the sun has to pass through the sun's atmosphere, which is colder, and our own atmosphere, before reaching the spectroscope. The bodies which it encounters in a state of vapour exercise their characteristic absorptions, and so produce the dark lines. Kirchhoff employed these lines to detect the presence of certain substances in the sun's atmosphere. Thus he found that it contains iron. For on observing the solar spectrum, and the line spectrum of iron side by side in the same spectroscope (as may be done by using the reflecting prism over half of the slit), he saw the bright lines in the latter exactly coincide with some of the dark lines in the former. In the same way, sodium and other metals are found in the sun's atmosphere. On the other hand, it may be inferred with tolerable certainty that certain terrestrial substances are wanting in the sun's atmosphere. Hydrogen is present; and it forms the extreme outside layer, as may be shown by directing the slit of the spectroscope to the extreme edge of the sun's disc, when the *bright* lines of hydrogen are seen.

**Doppler's Principle.**—When light falls on the eye or the spectroscope, the colour and the refrangibility depend upon the vibration-frequency, or number of vibrations arriving per second. Now, suppose a source of light to be in motion relatively to the spectroscope (say, moving towards it). The number of vibrations received per second will not be the same as the number emitted by the source, but in the case of relative approach it will be greater; just as the note received from the whistle of an approaching steam-engine is higher than when the steam-engine is stationary, and for the same reason. If V is the velocity of light, and $v$ the relative

velocity of approach or of retreat, the number of vibrations received per second will be equal to the number emitted by the source multiplied by the factor $\dfrac{V \pm v}{V}$.

If, then, there are absorption lines in the spectrum of a star, and it has a considerable velocity of approach or retreat relatively to the earth, the period of the vibrations in these lines will be altered, and they will appear to be slightly displaced from their usual places in the spectrum. The spectrum will give coloured parts exactly coinciding with the parts of the same colour of any other spectrum with which it may be compared, because colour is determined when refrangibility is. Only all the absorption lines will be seen to be a little displaced, that is, to correspond to slightly different colours. If, then, all the lines are nearly in the same positions as known bright lines, but all displaced a little in the same sense, we may infer that those lines are due to the presence of known substances in the star, and that their displacement is due to motion of the star. The amount of displacement enables a rough estimate to be made of the relative approach or retreat.

## CHAPTER XXII.

### *FLUORESCENCE. PHOSPHORESCENCE. ANOMALOUS DISPERSION.*

WHEN light falls on certain substances, such as sulphate of quinine or uranium glass, they exhibit a peculiar bluish colour, which seems to penetrate some way below the surface. This is called **fluorescence.** When light has produced this effect in a substance on passing through it, it cannot act in the same way again on more of the same substance. Light of certain qualities has been absorbed to produce the fluorescence, and the remainder has passed through. To examine what qualities of light are more effective for the purpose, a spectrum may be cast on a screen, and a piece of a fluorescent substance moved along the spectrum. Thus if a test-tube of sulphate of quinine is held in the light, and moved up the spectrum from the red end, it shows no peculiarity till the blue is reached, when the peculiar shimmer exhibited in sunlight appears, and increases as it is moved further along, remaining till the substance has been carried to some distance beyond the

extreme visible violet of the spectrum. Other fluorescent substances would show similar results. Thus it is the rays of the blue end and ultra-violet rays which produce fluorescence. In these experiments prisms of quartz should be used to produce the spectrum, because glass absorbs a good deal of the ultra-violet radiation.

To explain the cause of the phenomenon; consider that when radiation falls on a body, in general the effect is to heat it up, so that in a short time the body begins to radiate heat, that is, to emit radiation of long waves, longer than those which it has absorbed. In the same way, a fluorescent substance performs some sort of absorption of the rays of great refrangibility (short wave-length) that fall upon it, and emits them after transforming the radiation, as light of less refrangibility. This light, however, as the spectroscope would show, is not homogeneous.

The phenomenon of fluorescence has been applied to the study of the ultra-violet part of the solar spectrum. Sir John Herschel threw the spectrum on turmeric paper, and, in consequence of the fluorescence of the paper, observed a prolongation of the spectrum on the violet end. Dark lines, similar to the Fraunhofer lines, were observed in this part of the spectrum.

Connected with this is the phenomenon of **phosphorescence.** A substance which continues to emit light after light has ceased to fall upon it, is said to phosphoresce. The sulphides of calcium and strontium, in particular, phosphoresce for a very long time—several hours. It can be shown that phosphorescent light of a given quality has always been produced by light of shorter wave-length. This may be done by throwing a spectrum on a screen covered with phosphorescent material, and observing the colours when the light is cut off. It is the rays of high refrangibility that produce phosphorescence. The rays of low refrangibility cannot produce it, but if a substance is only faintly luminous by phosphorescence, these rays will excite it, and cause the substance to glow more brilliantly for a time, but to cease to glow altogether more quickly than if they had not fallen on it, that is, they hasten the emission of the phosphorescent light.

The character of the light emitted by a phosphorescent substance when illuminated by various qualities of light may be observed by the following experiment : Throw a horizontal spectrum (with vertical prisms) on a phosphorescent screen.

Look at this spectrum through a prism with its refracting edge horizontal. If the screen were not phosphorescent, each part of the spectrum, emitting light of a definite quality, would simply appear to be displaced vertically by a definite amount depending on the position of that part in the spectrum, that is, on the refrangibility of its light. But each part will emit light of refrangibilities equal to and less than that which falls on it. Thus each part will give a vertical spectrum limited on the side of greater refrangibility by light of the colour which falls on it.

In some substances which show phosphorescence the duration is very short. To study the time that it lasts, Becquerel has invented the phosphoroscope. The body is placed between two discs each having a series of apertures that comes opposite it as they rotate. Through the apertures in one disc it receives light, and through the others it can be observed; but it is hidden from view while light is falling on it. By suitably adjusting the discs, the interval between illuminating and observing the body can be made very small; and phosphorescence has thus been observed in a great many substances.

**Anomalous Dispersion.**—Most transparent substances, like glass, have higher refrangibility for light the shorter the wave-length, and so form spectra of white light in which the various colours have definite relative positions. For all substances, however, the refrangibility does not increase with the wave-length. Some substances produce spectra in which certain colours appear in unusual positions. Iodine vapour, for instance, has a greater refrangibility for red than for violet light. The anomalous dispersion of a substance may be observed by making it into the form of a prism and passing white light through it, and observing this light by means of a glass prism whose edge is held at right angles to that of the first. The light which in the spectrum formed by the first prism has an unusual position will be seen through the second to be broken away from the rest and to form an isolated patch.

The subject has been examined by Kundt. He finds that anomalous dispersion is shown by all substances showing **surface colours**, that is, appearing of a different colour by reflected from what they do by transmitted light. All such substances possess strong internal absorption for certain colours. The aniline dyes possess these properties. Kundt has found that, in such a substance, near to an absorption

band, the refrangibility of light just below, that is, on the red side of it, is abnormally increased, and that of light just above, on the blue side, is abnormally diminished.

# CHAPTER XXIII.

## *RAINBOWS. HALOS.*

THE phenomenon of the rainbow admits of a tolerably simple explanation on the principles of geometrical optics. It is caused by the sun's rays falling on rain-drops, and being deviated by them through reflexion and refraction to the eye of the observer. Each drop of water, in falling, takes, on account of the mutual attractions of its various particles, such a form that the area of its surface is the least possible. This form is that of a sphere. We shall have to consider, then, the action which a sphere of water will exert on rays of light which fall upon it.

Sometimes there are seen in the sky, besides the most prominent rainbow, others of greater angular radius and less brilliancy. It is the first-mentioned which we shall first consider. It is called the primary rainbow. It is formed by light which is refracted into the rain-drops, undergoes one internal reflexion, and is then refracted out again. An assemblage of parallel rays of light coming from the sun at S (Fig. 221), and falling on a spherical drop with its centre at O, will, after each undergoing one internal reflexion, proceed in various directions to points E, E', E'', etc. If there is a large assemblage of drops, they will all behave in the same way towards the light falling on them ; and to an eye at a fixed point, E, light will be sent along lines parallel to these directions from different drops. It appears, then, that the whole of the space occupied by drops capable of thus diverting the light to the observer's eye will seem to be illuminated. But we shall now see that much more light is sent in certain directions than in any others. For consider all the rays in the plane of the paper (supposed to contain the centre of the drop) which have a definite direction; that is, coming from S, and falling on the drop. Each of these undergoes, through two refractions and one internal reflexion, a definite amount of deviation from its original direction. The ray which meets the drop normally, at A, is deviated through two right angles. If we were to examine the

Y

paths of all the rays in turn meeting the surface T U V . . . of the drop, we should find that the deviations which they undergo get smaller and smaller until a minimum is reached ; and, then as we go on to Z, the deviations increase again.    Thus by the
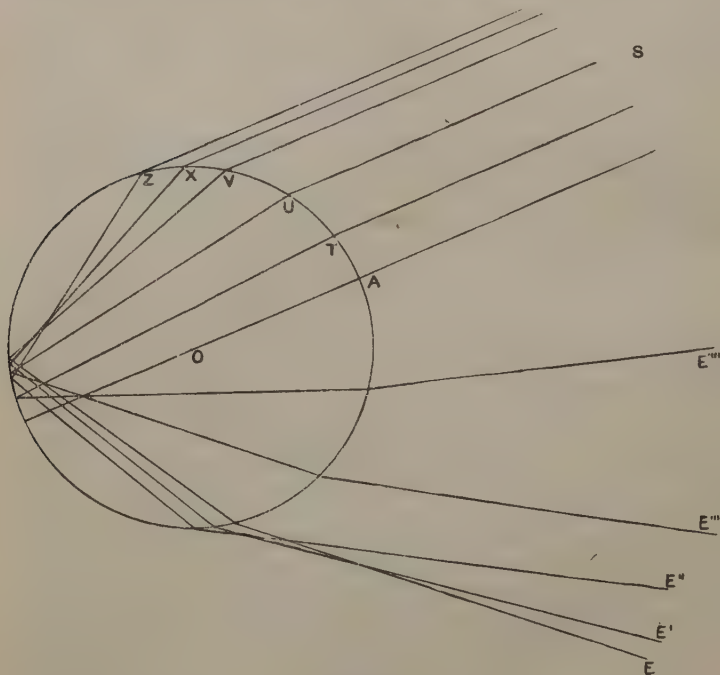


FIG. 221.

principle that when a quantity which is varying passes through a maximum or a minimum value, its variation is indefinitely small, it follows that a very large number of rays will emerge in or near to the direction of minimum deviation as compared with those in any other direction. The chief effect produced on the eye, therefore, is by the rays which undergo minimum deviation. If E denotes the position of an observer's eye, and O any drop such that the line S O, drawn from the sun, makes an angle with O E equal to the angle of minimum deviation, the observer will receive a considerable quantity of light from all drops situated in the same way as O. These drops all lie on the surface of a cone having a line through E parallel to

S O as axis, and a semi-vertical angle equal to S O E. Thus a circular arc of light will be seen, whose centre is at the point
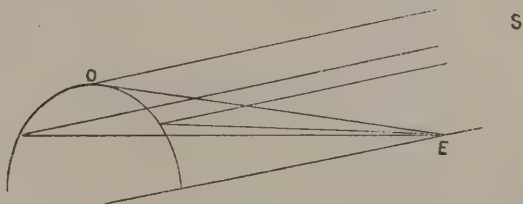


FIG. 222.

in the sky immediately opposite to the sun, and therefore below the horizon, and whose angular radius is equal to the angle of minimum deviation.

Hitherto no account has been taken of differences of refrangibility of the simple constituents of the sun's rays. But since the refrangibility varies from one colour to another, the minimum deviations, depending on refrangibility, will also be different. It is for this reason that the appearance is not a simple circle of white light, but a group of concentric circles of various colours. The minimum deviation increases with
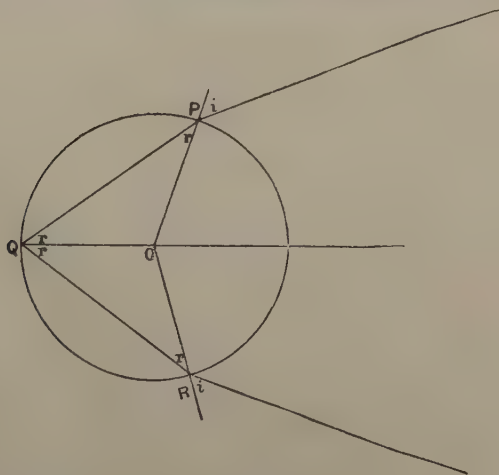


FIG. 223

the refrangibility, hence it is least for the red light and greatest for the violet. Therefore the angular radius of the red arc is

the greatest, and that of the violet the least; or, at the top of the arc the red is highest and the violet lowest.

The value of the minimum deviation for a given value of $\mu$, and for rays which have undergone one internal reflexion, may be found as follows: Let $i$ and $r$ denote the angles of incidence and refraction of any ray, and D the deviation of it. Then, by the accompanying figure, we easily see that—

$$D = \pi - POR + 2i$$
$$= \pi - 4r + 2i.$$

The small variation of this quantity must vanish for a small variation of $i$, where we have also the condition—

$$\sin i = \mu \sin r.$$

If, then, $d i$, $d r$ are corresponding indefinitely small variations of $i$ and $r$, we must have—

$$0 = -4dr + 2di,$$
$$\text{or } di = 2dr;$$
$$\text{and } \cos i\, di = \mu \cos r\, dr,$$
$$\therefore 2 \cos i = \mu \cos r.$$
$$\text{And } \sin i = \mu \sin r,$$
$$\therefore 4 \cos^2 i + \sin^2 i = \mu^2;$$
$$\text{or } \cos^2 i = \frac{\mu^2 - 1}{3}.$$

From this we can find $i$, and so $r$ and D. The angular radius of the corresponding arc which is seen is, of course, $\pi - D$.

The angular radii of the red and violet arcs thus found are, according to Glazebrook, $42° 2'$ and $40° 17'$.

If rays have undergone any number, $n$, of internal reflexions, we should easily find that the deviation is given by—

$$D = n(\pi - 2r) + 2(i - r)$$
$$= n\pi - 2(n + 1)r + 2i.$$

From this we can find, as above, the value of the deviation whose small variation vanishes as compared with that of $i$; that is, the maximum or minimum deviation, and hence that of the angular radius of the arc produced.

We have seen that light can reach the eye after one internal reflexion, and after undergoing any deviation between the minimum and 180°. Thus, although much more light is received after undergoing the minimum than any other amount

of deviation, yet some is received at other and larger deviations ; that is, from rain-drops lying within the bow ; and the space inside the bow appears rather brighter than that outside it.

The colours of the rainbow are not pure spectral colours ; for, on account of the finite angular diameter of the sun, the rays coming from it are not all parallel ; and so rays of a given refrangibility would not produce a single luminous circular line in the sky, but a number of such circles having their centres opposite to the various points of the sun ; and the aggregate of these would be a band of light of the form of a circular arc of a finite angular breadth. In the rainbow we have the superposition of such bands formed by the lights of various colours in slightly different positions ; and these overlap, or encroach on, each other.

The secondary rainbow is formed by light which has undergone two internal reflexions. Light which has undergone two internal reflexions is deviated by more than two right angles, and of this light the rays which undergo minimum deviation reach the eye in far greater number than any others. Here, too, the value of the minimum deviation increases with the
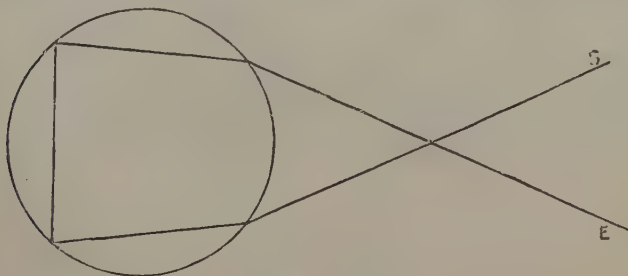


Fig. 224.

refrangibility, and that for the violet light is greatest, and for the red least. But, as these deviations are between two and three right angles, it follows that the emergent rays of violet light which have undergone minimum deviation make a greater acute angle with the rays from the sun than such emergent rays of red light. Hence in the secondary rainbow the violet ring is outermost, and the red innermost. The values of the minimum deviations of the red and violet rays, in this case, are $2\pi - 129° 20'$ and $2\pi - 125° 48'$. So that the angular radii of the arcs of red and violet light are $50° 40'$ and $54° 12'$.

Since some light reaches the eye at other deviations greater than these, it follows that the part of the sky outside this bow will be to some extent illuminated by light that has undergone two internal reflexions. Thus, from what we have already seen about the primary bow, the part of the sky between the two bows will be darker than that within the primary or without the secondary.

Of the other bows, formed by three, four, five, etc., internal reflexions, the third and fourth are in that part of the sky which lies towards the sun, and are thus rendered invisible by the sun's light. The fifth is formed away from the sun, but so much light is lost in five internal reflexions that it is seldom visible.

This is the elementary geometrical theory of the rainbow. It does not suffice to explain all the phenomena observed ; for the primary bow frequently appears accompanied by several others formed inside and close to it, and which rapidly become less and less brilliant as we pass away from the primary. These are called **supernumerary bows**. Their explanation was first attempted, on the principles of the wave theory, by Young, and has been fully given by Sir G. B. Airy.

In the elementary theory no account is taken of the differences in phase of the rays proceeding on emergence from a water-drop. We have supposed that their effects are merely superposed ; whereas we know that their combined effect will depend, not only on the rays themselves, but on their phase-differences as well. Now, if we consider the rays emerging from the drop in various directions, and



FIG. 225.

take account of their phases, it is found that, if O A is the direction of minimum deviation, a maximum of effect is produced in a direction O B, slightly above O A, and there is a series of other maxima, all smaller than the first, and rapidly decreasing, in directions O B, O C, etc., while below O A there is no maximum, but the effect rapidly falls off. For a given quality of light, then, the observer's eye is in a position to receive the first maximum of effect from certain drops, the second from others lying inside these, the third from others, and so on. And with the composite light from the sun we get the supernumerary bows, each of various colours, as well as the primary.

We see, further, that the full theory leads us to expect that the angular radius of the primary bow should be a little less than that given by the elementary theory.

These results of the wave theory have been verified by experiment by measuring the angular radii of the primary and supernumerary bows formed by a stream of water-drops.

### HALOS.

These are luminous appearances which are sometimes seen surrounding the sun or the moon. The complete phenomenon is represented in the accompanying figure, which, as well as the explanation of halos, is taken from Jamin's "Cours de



FIG 226.

Physique." It is very seldom that the full appearance, as here represented, is seen at once, or even a great part of it; it much more frequently happens that only a small part is to be seen. The phenomenon consists of (1) two circles, A A'; B B', with the sun or moon as centre, having angular radii of about 22° and 46°, and coloured, being red inside and violet outside; (2) a bright horizontal line, B A A' B', which is really a portion of a very large circle; (3) four spots of great brightness, at B, A, A', B', the points of intersection of this line with the two circles,—these spots are sometimes called *mock* suns or moons; (4) a bright vertical line, D C C' D'; (5) horizontal arcs touching the circles at D, C, C', D'.

The explanation of these appearances is due chiefly to Bravais. It is a much more simple explanation than, at first

sight, would appear to be necessary in the case of so complex a phenomenon.

The atmosphere sometimes contains, and especially in the cold mornings of spring and autumn, numerous tiny ice-crystals, whose shape is that of a prism, of cross-section a regular hexagon, and with end faces at right angles to the sides, as shown in the figure. It is the action of these prisms on the light from the sun or moon that produces the phenomenon.
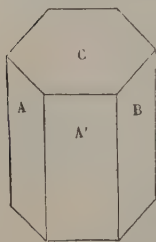
Light falling on one of these prisms may enter and emerge at two faces inclined at 60°, such as A and B, or at two faces inclined at 90°, such as A and C. And of the light acted on in this way much more proceeds in those directions for which the deviation is a minimum, in either case, than in any other direction, as in the case of the rainbows. The light undergoing minimum deviation between the faces inclined at 60° produces the smaller circles; and light undergoing minimum deviation between the faces inclined at 90° produces the larger circle. The colouring is due to the fact that the refractive index, and therefore also the minimum deviation, is greater for violet than for red light.

Fig. 227.

To explain the horizontal and vertical lines, we must notice that the prisms, in falling, tend on the whole to set themselves in the positions in which the least resistance is offered to their motion by the air. Now, among them there will be many which are very long as compared with their breadth, and these will tend to keep positions in which their side faces are vertical and their end faces horizontal; and there will also be many which are very short as compared with their breadth, which are, in fact, hexagonal plates in shape; and these will tend to keep their end faces vertical. Thus the prisms will present many vertical and many horizontal faces; the former of these, by reflexion of the light, produce the line B A A′ B′, and the latter, also by reflexion, the line D C C′ D′.

From the points A, A′, B, B′ a great quantity of light proceeds. At these points the circles and the horizontal line reinforce each other. But this is not all. Anywhere in the circles A A′, B B′ the light has undergone minimum deviation by refraction at two faces of a prism. And there will be a very large number of pairs of faces inclined at 60°, with refracting edge vertical, in a suitable position to produce the

spots A, A'; and a very large number of pairs inclined at 90°, in a suitable position to produce the spots B, B'.

To account for the curves $a\,a$, $a'\,a'$. There is a great number of plates with their hexagonal faces vertical; and thus presenting pairs of faces inclined to each other at 60° with their refracting edges, or the lines in which the pairs would meet if produced, turned in all horizontal directions. The rays refracted at the pairs of faces in planes orthogonal to the refracting edges give rise to the light from the parts C, C'. The rays which pass in sections oblique to the refracting edges are more deviated than those in orthogonal sections, and are also turned away further from the vertical planes in which they were before meeting the crystals. This accounts for the curvatures of the bright strips $a\,a$, $a'\,a'$.

To account for the curves $d\,d$, $d'\,d'$. There is a large number of prisms with end faces horizontal and sides vertical, thus presenting pairs of faces inclined at 90° with refracting edges turned in all horizontal directions. The rays refracted at these pairs of faces, some in sections orthogonal to the refracting edges, some in sections more or less oblique to them, give rise to the curves $d\,d$, $d'\,d'$.

## CHAPTER XXIV.

### *THE EYE.*

THE human eye is a ball of nearly spherical shape, capable of being turned about in its socket. Its outer shell, D, is composed of a hard white substance, and is called the **sclerotic coat.** The part of this that is seen between the eyelids is what is called the " white of the eye." In the front of the sclerotic is an aperture to admit the light, which is closed by the hard transparent substance P, called the **cornea**; this is in shape like a very convex watch-glass, and its material resembles horn. The sclerotic is lined with a thin membrane called the **choroid coat,** which is made very black and opaque by means of a pigment which saturates it. Through the sclerotic and the choroid the optic nerve, N, enters the eye from the brain.

Inside the choroid is the portion of the eye which is sensitive to light. This is the **retina.** It is a delicate, almost transparent, membrane, in which the numerous fine filaments of the

optic nerve spread out and end by being connected, first with ganglion-cells like those of the brain, and next with a number of minute bodies called **rods** and **cones**. The rods are minute cylinders, and the cones somewhat thicker and in shape like



FIG. 228

a flask. These are all closely packed on the outer surface of the retina and perpendicular to it, and form there what is some-times called **Jacob's membrane.** It is on them alone that the light produces a direct action after traversing the other parts of the retina.

The retina possesses the property of continuing to feel the effect of light which has acted upon it for a short time after the light has been removed. This property is generally spoken of as the *persistence of impressions.* It may be illustrated by many experiments. If a stick be taken from the fire with the end glowing, and whirled rapidly round, a bright circle, and not a single bright spot changing its position, will be perceived. Maxwell's colour-top for combining colours by making them the sectors of a rotating disc depends on this property. The duration of the sensation after the light has been cut off from a given part of the retina depends on the intensity of the light ; it may amount to a considerable fraction of a second.

In the front of the eye is the **crystalline lens,** C. The function of this is to throw a real image of an external object on the retina. This lens is double convex, being more curved behind, or towards the inside of the eye. It is not homo-geneous, but its density and refractive index increase from the surface to the middle. This tends to diminish its spherical aberration, since, in a homogeneous lens with spherical surfaces,

the marginal rays are more convergent than the central ones. At the same time, aberration of the marginal rays is by no means entirely absent. This may be shown by the following simple experiment : Let an object, such as a printed page, be brought just too near to the eye for it to be seen distinctly. If now a piece of paper pierced with a pin-hole be placed close in front of the eye, the part of the object in the centre of the field of view may be seen quite distinctly with all its details, although it will appear less bright than before. Hence the rays which come from any point of it and pass through the pin-hole are brought to a true focus on the retina, although all the rays from this point which reached the lens before could not be so focussed. It follows that the rays which pass through the middle of the lens are more convergent than those which pass through the marginal portions.

In front of the crystalline lens is the **iris**, I I, a diaphragm pierced with a circular aperture called the **pupil**, which admits light to only a portion of the lens close round its optic axis. The size of the pupil accommodates itself to the brightness of the objects looked at ; but this action is not immediate,—it takes some seconds for it to be accomplished. If one goes from bright sunlight into a comparatively dark room, objects may be at first invisible, and after a little while become pretty easily discernible as the pupil dilates to suit the altered conditions of light. On the other hand, when one goes from a dark room into the sunlight, things appear at first to be of a dazzling brightness which is painful to the eyes.

The pupils of some animals, such as cats, consist, in bright light, of narrow vertical slits, which are capable of dilating considerably in the dark, or if the animal is alarmed or otherwise excited. This explains the facility with which these animals see in places where there is very little light.

The crystalline lens has the power of *accommodation* for various distances of the object looked at. This is accomplished by alteration of the curvatures of its surfaces, and hence of its focal length. The alteration of curvature is effected by means of a muscle, A A, going round the rim of the lens, and called the **ciliary muscle**. The nearer an object is to the eye, the shorter has the focal length of the lens to be in order to form an image on the retina ; hence the greater must the curvature be, and the more must the muscle be contracted. In the case of a normal eye, when the muscle is not in action, the eye is adapted for viewing infinitely distant objects, or for bringing a parallel pencil of light to a focus on the retina.

Suppose O to be a bright point, too near to the eye to be seen distinctly. Let a piece of paper be pierced with two pin-holes, *a*, *b*, and held between O and the eye ; *a* and *b* being near enough together to allow light from O to enter the pupil
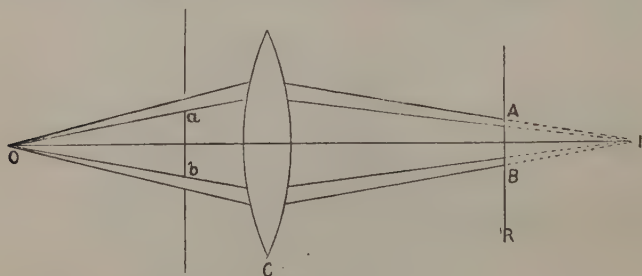


Fig. 229

through both at the same time. Now, the crystalline lens, C, would form an image of O behind the retina, R ; and if the paper screen is not used, the rays going to form this image will meet a large part of the retina, and O will not be seen distinctly. But when the screen is interposed, two very narrow pencils from O meet the retina in two places, A and B, of very small extent, and two pretty distinct images of O are seen. A small bright object, such as a pin-head on which the light is shining, may serve as object, and two images of it will be seen. If, instead of two holes in the screen, we pierce several, just as many images of the object will be seen. If the object is gradually moved away, the images approach each other, and at last coincide when the object is at the least distance of distinct vision from the eye. I then is formed on R.

Between the cornea and the crystalline lens is a transparent fluid called the **aqueous humour**. And in the ball of the eye, between the crystalline lens and the retina, is a transparent jelly called the **vitreous humour**.

In the retina, almost opposite the middle of the crystalline lens, is a spot called the **yellow spot**, in the centre of which there is a slight depression called the **fovea centralis**, or pit of the retina. It is here that vision is most exact ; and when an object is directly looked at, the eye is turned so that the image of the object is formed on this depression. Here the cones are smaller and more closely packed ; and no nerve-filaments or blood-vessels pass across the pit. The distance between two points can be distinguished, if looked at directly, when they subtend an angle of 1′ at the eye; for this corresponds

to the distance between two cones in the yellow spot. Thus two visible or bright objects, an inch apart, and 100 yards distant, can be distinguished from each other.

The yellow spot is less sensitive to weak light than the surrounding parts of the retina. For this reason, if stars of small brilliancy, such as the Pleiades, are looked at a little obliquely, they produce a more brilliant effect than if viewed directly.

The part of the retina where the optic nerve enters the ball of the eye is not sensitive to light, because there the rods and cones are wanting. This spot is called the **blind spot.** It is situated a little to the inside of the centre of the retina, or yellow spot. The blind spot is about large enough to conceal an object an inch in diameter held at a distance of ten inches from the eye. It may be recognized by the following experiment: Make a black mark—a circular black spot of one inch diameter, say; hold this at about a foot from the eye, and then run the eye along the paper from the mark horizontally inwards,— that is, if using the left eye, to the right; if the right eye, to the left. When the eye reaches a point about three inches from the spot, this disappears, and comes into view again on looking still further along.

Besides the blind spot, there are other parts of the retina which do not perceive light. This is caused by the blood-vessels crossing the retina.

It may be thought that the blind spot would form a serious inconvenience; but, on the contrary, its existence is not, as a rule, perceived, and can only be recognized by some such experiment as that which has been described. Indeed, it was not known that such a spot existed until Mariotte made special experiments to determine the condition of the eye as regards sensitiveness to light at the place where the choroid and Jacob's membrane are wanting. This he did to ascertain whether it is the choroid or the retina that perceives light; but, of course, we know that such experiments would be inconclusive, because at this spot the sensitive portion of the retina is absent as well as the choroid.

From what has gone before, we see that the picture of external objects which is formed on the retina is clear and distinct in all its details only in one very limited portion, namely, the part formed on the yellow spot. The rest is, as it were, only roughly indicated; and has besides one large gap in it corresponding to the blind spot, and several smaller ones. Hence there can be obtained at once a clear impression of

only a very limited portion of what is looked at, that portion, namely, towards which the gaze is directed. Of the rest, some is seen indistinctly, and portions not at all. But it is the portion towards which the eyes are directed that the attention is for the time concentrated upon, and that portion is all that we wish to see distinctly at once. What we see of the rest is enough to give us an idea of how the part we are gazing at is situated relatively to it. And that the whole is not seen with equal distinctness at once is amply made up for by the facility with which the eyes can be turned from point to point. The gaps in the picture are no impediment to vision, for, as we have seen, their very existence cannot be recognized without the help of special experiments. Further, it is possible that the mere muscular exertion, slight as it is, of turning the eyes from one point to another, assists in directing the attention to a new object.

**Astigmatism.**—A small pencil of light entering the eye is acted on by the optical arrangement formed of the cornea, the crystalline lens, and the aqueous and vitreous humours; and such a pencil will only be brought to a true geometrical focus in the vitreous humour, or at its boundary, that is, on the retina, if this arrangement is symmetrical about an axis, and the pencil passes along it; otherwise it gives rise to an astigmatic pencil with two focal lines formed at different distances behind the crystalline lens. Now, in most eyes there is some want of symmetry in the optical arrangement; the curvatures of the cornea and lens may be different in different directions, or there may be some want of centring of the cornea with respect to the lens. In the normal eye the errors are not sufficient to produce any appreciable effect. But if these errors are excessive, indistinct vision is caused. The most obvious practical result of this defect is that two intersecting rectangular straight lines (namely, those parallel to the focal lines of the astigmatic pencil produced) could both be seen distinctly, but not at the same time.

To remedy this defect, cylindrical spectacles are employed, in which the surfaces of each lens are portions of large cylinders with axes parallel to each other. Such a lens would render a pencil with a point-focus astigmatic. If a lens, then, is selected capable of producing the proper amount of astigmatism, and held before the eye in the proper position (that is, with the axes of its surfaces parallel to the more distant focal line of the astigmatic pencil that would be formed in the eye), it will produce such a degree of astigmatism in an

incident pencil that this will give rise to a pencil with a point-focus in the vitreous humour.

**Chromatism.**—The eye is not free from chromatic aberration; but this defect is not perceived with ordinary light, because the rays of intermediate refrangibility are so much brighter than the extreme rays. It can, however, be demonstrated by the following experiment described by Helmholtz: Take a piece of glass coloured with cobalt oxide. This cuts off the middle part of the spectrum, and lets through only red and blue light. Look through this at a bright light, such as a street-lamp, from a distance. There will be seen a red flame surrounded by a bluish-violet halo.

The chromatic aberration of the eye may also be recognized by the following simple experiment: Look at a narrow bright object, such as a gas-flame seen edgeways from some distance. Now place a screen (the hand will do) close in front of the eye, with its edge parallel to the object, and so as just not to conceal it. This will cause the light from the object to enter through the side of the pupil only. The rays will then be deviated by different amounts—the blue most, the red least. The blue rays will go towards the side of the retina on which is the image of the edge of the object turned away from the screen; and this edge will appear to be coloured blue. The other edge will be coloured red.

The same thing can be shown in a slightly different manner by looking at a window bounded by a straight-edge, and holding a finger close in front of the eye, and so that the image of its edge comes close up to that of the edge of the window. A bluish or a reddish tinge will be perceived along the edge of the window, according as the finger is held so as nearly to cut off the view of the window or of the wall bounding it.

The image, formed on the retina, of an object looked at is an inverted one; but it is the impression of an erect image that is always conveyed to the brain. This has frequently been felt as a difficulty in the theory of vision. But it is to be observed that the picture on the retina gives an accurate impression of the relative distances from each other of the parts of the object looked at and the parts of the observer, such as his hands, which are to be seen in the field of view; and it is a matter of experience which part of the picture corresponds to above and which to below. The retina has learnt by experience where to expect to feel the effect of the upper part of an object, and where that of the

lower. Further than this, so far as we know how the impressions are conveyed from the retina to the brain, there is no more reason to expect the impression of an inverted picture than that of an erect one.

When one looks at an object, the crystalline lenses must be focussed for the distance of the object, and the optic axes of the two eyes must be directed towards it, the degree of convergence of the axes increasing with the nearness of the object. These two circumstances afford the means of forming an estimate of the distance of the object, when the distance is not very great; but, for great distances, both the focussing of the lens and the convergence of the eyes are practically *nil*, and thus do not assist in forming an estimate of the distance. Of course, it frequently happens that other circumstances help one in judging a distance, such as seeing objects of approximately known dimensions in the field of view.

Here may be mentioned the errors sometimes made with regard to the sizes of object, when circumstances lead to their distances being wrongly estimated. Thus in a clear mountain atmosphere, distant mountains are thought to be nearer than they are, and consequently are judged to be smaller than they are. And objects seen in a fog sometimes appear to be of enormous dimensions, because, being seen indistinctly, they are thought to be further off than they are.

**Binocular Vision.**— If we look at a group of objects with one eye, the appearance perceived is very much like that of a flat picture of the object, only that the eye can form some estimate of the distances by means of the focussing of the crystalline lens. But if both eyes are used, the case is quite different. We now appreciate much more clearly the **solidity** or **depth** of the field; and this is the great advantage of vision with two eyes over vision with one. It must be noticed, in the first place, that, though a picture is formed on each retina, yet only one impression is conveyed to the brain. To each point in one retina there is a **corresponding** point in the other; and when the two images of an object are formed at corresponding points, as is in general the case, the pictures produced overlap, and only one is perceived. Now, in looking at a solid object with both eyes—for instance, the hand held edgeways in front of the face—it is not quite the same picture that is perceived by the two, because they see it from different points of view. But if it were a flat picture of the object that is viewed, the two appearances presented would be the same (except for slight differences of the distances of the various

parts). Hence, if we consciously compared the pictures formed on the two retinæ, we should infer the solidity of the object; and this is the impression unconsciously conveyed to the brain. When the two eyes are directed towards one of the objects in the field, there will be formed double images of other objects, that is, images not at corresponding points of the retinæ. If a finger is held up before the wall, to the two eyes it will be seen to cover two different parts of the wall; or, if the wall be looked at, two indistinct and shadowy images of the finger will be observed before different parts of it. This, however, forms no obstacle to distinct vision: the object which is directly looked at forms a single picture for the brain. The pictures of the other objects are formed on parts of the retina where they are not so distinctly seen; and their being double merely creates the impression of solidity in the whole field of view.

The **stereoscope** is an apparatus by means of which the combined action of the two eyes in viewing solid objects is well illustrated. Two photographs of an object or group of objects, are taken from slightly different points of view—the positions that the two eyes may occupy in looking at it. These are then viewed, one by each eye, so that each eye
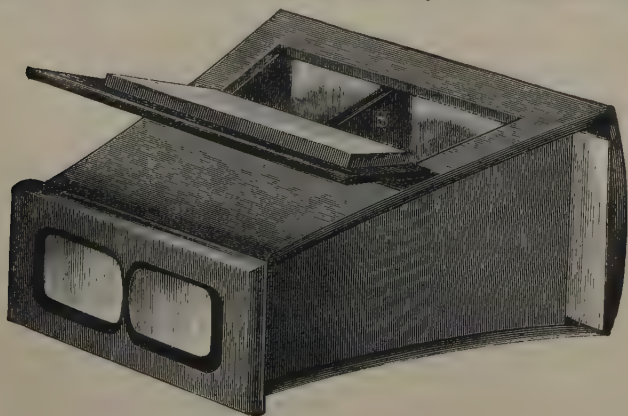


Fig. 230.

obtains just the same view that it would have of the actual object. The essential part of the stereoscope which enables this to be done properly consists of two prisms of small refracting angle, which are placed before the eyes with the

z

refracting edge turned inwards, so that each photograph appears somewhat displaced towards the other, and the images of the two, as seen through the two prisms, overlap; also the surfaces of the prisms are slightly convex, so that they also act as convergent lenses, and produce magnified images of the photographs at a greater distance than these from the eyes. The accompanying figures represent the stereoscope and a cross-section of its two prisms.

If the stereoscopic views of a solid geometrical figure, such as a cube, be placed in the apparatus, so that the view for
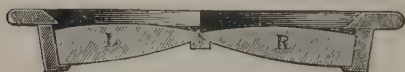


FIG. 231.

the right eye comes before the left, and *vice versâ*, then the impression obtained will be that of a hollow figure, for each eye then actually gets the view it would have if the two looked directly at a hollow figure. A solid cube standing on a flat surface would appear like a hollow cubical recess let into the surface.

When the normal eye is at rest, that is, when the ciliary muscle is not acting to increase the convergence and diminish the focal length of the crystalline lens, it is focussed for infinite vision, or parallel rays of light coming along the optic axis will be brought to a focus on the yellow spot. And such an eye can with ease bring to a focus on the yellow spot a pencil of light coming from a point at any distance between infinity and about ten inches before the eye. That is, the eye can see distinctly objects at all distances between these limits ; that is to say, the images formed are distinct, and not blurred, but they will, of course, if the objects are too far off, be too small for details to be distinguished. When an eye cannot adjust itself for viewing objects at such distances, vision must be regarded as defective. The defects are of two sorts, according as the eye cannot see objects which are far off or those which are close to it.

Some eyes cannot bring a parallel pencil to a focus on the retina. Even when the eye is quite at rest, its focal length is too short, and the focus of such a pencil is formed in front of the retina ; or we may say that the retina is too far away from the lens. This defect generally diminishes with advancing age. It is called **myopia**, or **myopy**, or, in ordinary language, "short-sightedness." To correct it, we must notice that the eye can only adapt itself for tolerably divergent pencils ; and for viewing a distant object, that is, for receiving pencils

of little divergence, a divergent or concave lens must be used. Hence the use of concave spectacles and eye-glasses by the short-sighted. An eye having this defect can generally see distinctly objects much closer to it than the least distance of distinct vision for normal eyes.

Some eyes cannot see distinctly objects which are near. For them the difficulty arises that when an object (such as a printed page) is held far enough off to form a true image on the retina, this image is too small for its details to be recognized. Such an eye is incapable of bringing to a focus on the retina a pencil coming from a point at the ordinary least distance of distinct vision. The focal length cannot be made short enough by the action of the ciliary muscle. This defect generally comes on or increases with advancing age. It is due to a hardening of the crystalline lens ; so that the ciliary muscle is unable to give it sufficient curvature and a small enough focal length. It is called **presbyopia, presbyopy,** or **presbytia,** or, in ordinary language, " long-sightedness." To correct it, notice that the eye cannot adapt itself for a pencil of much divergence ; thus these pencils must be made less divergent, and a converging or convex lens must be used. Hence the use of convex spectacles by the long-sighted.

For a person short- or long-sighted, if we know the greatest or the least distance of distinct vision, it is a matter of simple calculation, from the formula for thin lenses in combination, to calculate with sufficient accuracy the focal lengths of the spectacles required—in the first case to produce a combined focal length equal to infinity, and in the second equal to about ten inches, or whatever distances may be required.

### Examples.

1. A screen with a small hole in it is held before the eye, within the least distance of distinct vision, and against a good light. Between this and the eye, and very close to the eye, is held a very small object, such as a pin-head. A large, blurred, and *inverted* image of the object is seen. Explain this, giving a diagram.

2. The greatest distance at which a short-sighted person can see distinctly is 2 feet. What spectacles does he require so as to see objects at very great distances ? and if his least distance of distinct vision without the spectacles was 6 inches, what will it be with them ?

3. Find what spectacles must be used by a person who cannot focus his eyes on an object nearer than 2 feet so as to be able to read distinctly at 10 inches' distance.

## CHAPTER XXV.

### *RELATIONS BETWEEN LIGHT AND ELECTRICITY.*

WE shall now point out some of the relations that exist between magnetic and electric actions on the one hand, and light on the other.

Faraday discovered that if plane polarized light is passed through a transparent substance in a magnetic field, the plane of polarization is, as a rule, rotated. The rotation is greater the smaller the angle between the direction of the rays and the lines of force, and disappears when these two directions are at right angles. This action on light may be shown in the following manner : A strong electro-magnet is furnished with pole-pieces having holes through them in a line with each other so as to allow the light to pass. Between the poles is placed a piece of very dense glass, and two Nicols are placed in the path of a beam of light passing through the two holes and the glass, one on each side of the arrangement. Before making the current in the electro-magnet, the position of the second Nicol is found for which the light coming from the first is quenched. If, then, the current is established so that the glass is in a magnetic field, the light will reappear, and the second Nicol must be rotated to again extinguish it. The sense in which the plane is rotated depends upon the positive direction of the lines of force of the field. If the field is reversed, as may be done by reversing the current in the electro-magnet, the sense of rotation is reversed. Thus for light travelling along the lines of force one way the rotation is right-handed, and if it travels in the opposite way the rotation is left-handed. In this way a substance in a magnetic field behaves differently from an ordinary rotating substance such as quartz. From this property it follows that the rotation produced in a beam can be increased by reflecting it to and fro along the lines of force of the field. For suppose the beam is passed along the lines of force, and then reflected back and along again. If the rotation in the first path is right-handed, that in the second is left-handed, and therefore in the same sense *in space* as the first, and that in the third is again right-handed ; so that the entire amount of rotation is three times as much as for a single passage through the medium. If the beam were reflected to and fro through a rotating substance such as quartz, the rotations would cut each

other out two and two, and the entire amount of rotation produced in the end, there being an odd number of passages, would be only that due to a single passage.

Verdet has investigated the laws of the rotation of the plane of polarization in a magnetic field, and has found that it depends on the substance through which the light passes, and is proportional to the length that it traverses, to the intensity of the field, and to the cosine of the angle between the direction of the rays and that of the lines of force of the field. Putting the three last quantities together, we may say that the rotation produced between any two points in a ray is proportional to the difference of magnetic potential between those points.

Another action of magnetism on light was discovered by Dr. Kerr. If plane polarized light be reflected from the polished pole of a magnet, the plane of polarization is rotated. This may be shown by using an electro-magnet. If, with the current off, the position of the plane of polarization of the reflected beam is found, then when the current is started the plane is found to be rotated.

Dr. Kerr also discovered that if a transparent insulating medium be subjected to a strong electrical stress, it becomes doubly refracting. This may be shown by passing a beam of plane polarized light through the medium in a direction at right angles to the electrical lines of force, when the beam will emerge elliptically polarized.

Dr. Hertz of the University of Bonn, and other physicists, have performed numerous experiments on electrical vibrations, which show that these vibrations behave in many respects like those which constitute light. Consider first how such vibrations can be obtained. Suppose two oppositely charged electrical conductors to be brought near to each other; the space between and surrounding them is in a state of electrical stress. If the electrical difference of potential between the conductors be suddenly destroyed by discharging them to each other, the electrical stress is removed, and a wave of electrical displacement passes out into space, just as if a stretched string is pulled a little on one side at a certain point, and then this point suddenly let go. When the conductors are discharged to each other, however, this is not all that happens, but the electrical charge surges to and fro from one to the other, making several oscillations, gradually decreasing in intensity till the potentials of the conductors are equalized. In consequence of this a series of waves of

electrical displacement passes out into space. The period of these waves, or the frequency with which they pass any given point in space, will depend upon the size, form, and general arrangement of the conductors, the period being the same as that of the oscillations of the charges on the conductors. Now, if the conductors are continually charged and discharged, as may be done by connecting them to the terminals of an induction-coil, such series of waves follow each other in rapid succession. The waves being obtained in this way, the next step is to have some suitable apparatus for detecting their presence at any point in space. This is called a *receiver*. The one usually employed by Hertz was a ring of wire with a small air-gap, the size of the ring being adjusted so that the period of oscillation of a charge from one side of the gap to the other round the ring is the same as that of the *vibrator*, consisting of the arrangement of two conductors with a *spark-gap* between them. When the receiver is suitably placed and the size of the gap adjusted, it will show sparks on the passage of waves past it from the vibrator.

The electrical waves will be reflected if they are made to fall on a conducting surface as on a sheet of metal fastened to the wall; and by the help of this phenomenon the wave-length can be determined. For if the receiver is placed in the way of the direct and reflected waves, it is found that at certain points there is no sparking. This indicates a state of stationary vibration of which these points are the nodes, the distance between two consecutive ones being half of a wave-length. Again, the vibration-period is determined from the known constants of the apparatus. Thus the velocity of propagation is found.

Other experiments have been made by Hertz and others, also depending on states of stationary vibration, to determine the velocity of propagation of electric disturbances along a wire.

The best of these experiments have shown that electrical disturbances are propagated both in air and through wires (the propagation through a wire being really that through the space surrounding it) with a velocity which is, within the limits of experimental errors, the same as the velocity of light.

We have noticed that the vibrations can be reflected from metallic surfaces; and, by means of a curved mirror formed of sheet-metal, the spark-gap of the vibration being situated in the focus, the waves can be concentrated or gathered up along a definite direction, as in the case of light.

Again, by using a very large prism of pitch, Hertz found that the waves could be refracted through this substance, it being *transparent* to them, in the same way as light-waves are refracted through glass.

Experiments indicate that the electrical vibrations are transverse to the line along which they travel, like those of light; and indeed, since these vibrations are found to travel along a line at right angles to the spark-gap, the manner in which they are generated indicate that they are transverse, for the action by which they are produced at the spark-gap is one which takes place at right angles to the line of propagation, and not in the line of propagation, as when sound-waves are generated. The receiver works best with its spark-gap parallel to that of the vibrator, and shows no sparks at all when the two gaps are at right angles. This indicates that the action has reference to some plane through the line of propagation, and is not the same relatively to all such planes. Again, if a grating of parallel wires be placed in the way of the waves, it will produce no effect on them when the wires are at right angles to the spark-gap, but will intercept and reflect them when the wires are parallel to the spark-gap.

All these experiments show a striking resemblance between light-waves and waves of electrical disturbance; and it is extremely probable that the two are just the same in kind, the only difference between them being the enormously greater frequency of vibration and shorter wave-length in the case of the light-waves than in the case of such electrical waves as have hitherto been produced by such experiments as we have noticed. The waves transmitted from a spark-gap may be compared with those of light in which the vibrations are all in the same direction; that is, with those of polarized light.

The velocity with which light-vibrations are transmitted through a medium is inversely proportional to its index of refraction, $\mu$; that is, the index of refraction for vibrations of the given period. Let us consider on what the velocity of propagation of electric vibrations will depend. The velocity of propagation through an elastic medium of a disturbance which consists of a displacement of part of it relatively to the neighbouring parts, is given by the formula $V = \sqrt{\dfrac{E}{D}}$, D being the density of the material displaced, and E the elasticity which is called into play by the displacement. The elasticity which acts in the propagation of a transverse displacement is the *rigidity*. In the given case of electric vibrations the elasticity

is the electrical stress acting per unit of electrical displacement transversely to the line of propagation of the vibrations, or in the direction of the spark-gap. This quantity varies from one medium to another according as the electrical inductive action for given electrical forces varies; that is, according as the specific inductive capacity varies. Now, according to Maxwell's theory, the specific inductive capacity, K, of a medium is simply proportional to the electric displacement, or strain, produced by a given stress. Thus $E \propto \dfrac{1}{K}$.

And therefore $V \propto \dfrac{1}{\sqrt{K}}$. Hence, as we pass from one medium to another we should expect to find, according to this reasoning, that $\mu$ is proportional to $\sqrt{K}$. Now, the constants $\mu$ and K for the standard medium, air under standard conditions, are both taken to be unity. Hence the relation between them for any medium whatever is $\mu = \sqrt{K}$.

Maxwell supposed light to be an electro-magnetic action, and he gave the relation $\mu = \sqrt{K}$ before the existence of electric waves was demonstrated by experiment. The agreement between the values of $\mu$ and $\sqrt{K}$ for different media is not very close, although a certain amount of correspondence is found between them. The discrepancy is to be explained by the fact that for most media $\mu$ depends upon the wavelength, and it can only be experimentally determined by optical methods for very short wave-lengths. If its limiting value could be found—that is, for wave-lengths which are practically infinitely great, there would no doubt be then a much closer agreement, since for very long waves in any medium the velocity of propagation should be proportional to $\sqrt{E}$.

# APPENDIX

## On the Relation between the Brightness of an Object and that of its Image.

If an object situated in a medium of refractive index $\mu$ forms an image by means of rays refracted through any number of surfaces and finally into a medium of index $\mu'$, the luminosities, I, I', of object and image, satisfy the relation—

$$\text{I} : \text{I}' = \mu^2 : \mu'^2,$$

supposing no light to be lost by reflexion at the surfaces or by absorption in the media.

This is generally proved, in the case where the pencils are not all very narrow and along an axis of revolution of the refracting surfaces, by showing that if this relation did not obtain, we could get an image brighter than the object by viewing this through a suitable combination of lenses—a result which would be contrary to experience. The following direct proof may, however, be given for pencils of any sort :—

We notice first that if a small luminous surface sends a certain quantity of light to a second small surface, for the second to send the same quantity of light to the first it must be equally bright with it. For if $s_1$, $s_2$ are the areas of the two surfaces, $I_1$, $I_2$ their luminosities, $d$ the distance between them, and $\theta_1$, $\theta_2$ the inclinations of their normals to $d$, the lights which they send to each other are—

$$\frac{I_1 s_1 s_2 \cos \theta_1 \cos \theta_2}{d^2}, \text{ and } \frac{I_2 s_1 s_2 \cos \theta_1 \cos \theta_2}{d^2}.$$

And for these to be equal, $I_1 = I_2$.

Now let $s$ be a small area of an object in a medium of refractive index $\mu_1$, and let it send out a sheaf of rays which is refracted across the surface R T into a second medium of index $\mu_2$. The cone of rays from $s$ converging to the point R is refracted into a cone of different solid angle. If $i$ and $r$ are the angles of incidence and refraction at R, the angular breadths of these solid angles in the plane of refraction are $di$, $dr$; and their angular breadths at right angles to this plane are proportional to sin $i$, sin $r$.

Now—

$$\mu_1 \sin i = \mu_2 \sin r,$$
$$\therefore \mu_1 \cos i \, di = \mu_2 \cos r \, dr.$$

And if the solid angles are $\omega_1$, $\omega_2$—

$$\frac{\omega_1}{\omega_2} = \frac{di \sin i}{dr \sin r} = \frac{\mu_2^2 \cos r}{\mu_1^2 \cos i}.$$

Now let $I_1$ be the luminosity of $s$, and $I_2$ that of the surface R T as seen in the medium $\mu_2$ by the light coming originally from $s$.
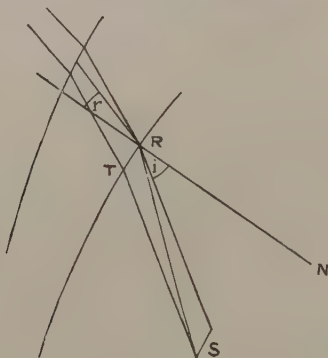


Fig. 232.

$I_1$ is the luminosity R T must have to send back to $s$ the same amount of light as $s$ sends to it—that is, as R T sends on into $\mu_2$. If $s_1$ is the area of the surface R T, the quantities of light received and sent on by it are—

$$I_1 s_1 \omega_1 \cos i, \text{ and } I_2 s_1 \omega_2 \cos r.$$

And these are equal. Thus from above—

$$I_1 : I_2 = \mu_1^2 : \mu_2^2.$$

In the same manner, the luminosity of the second surface across which the rays pass, as seen in the third medium of index $\mu_3$, is proportional to $\mu_3^2$; and so on.

If the rays on passing across the last surface form a true image of $s$, so that the combination of surfaces is aplanatic for $s$, the brightness of this image is the same as that of the last surface; for it subtends the same solid angle at the point of view as the portion of the surface which sends the same quantity of light to the point of view. Thus, if $\mu$, $\mu'$ are the refractive indices of the medium in which the object is and that in which the eye is which sees the image, and I, I' the luminosities of the object and image—

$$I : I' = \mu^2 : \mu'^2.$$

# ANSWERS TO EXAMPLES

## I.

1. $2\frac{1}{2}$ feet ; $3\frac{1}{2}$ feet.
2. 50 cms.
4. 27·5 nearly.
5. Inclined to the line joining it to the point at an angle whose sine is $\dfrac{\pi}{4}$.

## II.

3. 3·92 cms. about.
4. $6\frac{2}{3}$ feet.
5. $-9\frac{9}{11}$ inches.

## III.

1. 3·37 inches nearly.
2. 1·5613 about.
4. The centre of the globe, its real position.
5. At distance $\dfrac{875\mu - 32\mu^2 - 75}{64\mu^2 - 47\mu + 3}$ from same face.
6. $1\frac{1}{5}$ inches.

## V.

13·7 and 76·7 cms. about.

## XII.

0·0006 cms.

## XXIV.

1. Focal length 2 feet ; 8 inches.
2. Focal length $-17\frac{1}{2}$ inches.

# INDEX